A Concise Summary of Continuum Mechanics

Condensed notes of the material covered in the third-year undergraduate course *Mehanika kontinuov* (Continuum Mechanics) at the Faculty of Mathematics and Physics at the University of Ljubljana in the academic year 2020-21. The course covers continuum mechanics at an upper-undergraduate level and draws heavily from Landau and Lifshitz's *Theory of Elasticity* and *Fluid Mechanics*.

Disclaimer: This document will inevitably contain some mistakes—both simple typos and legitimate errors. Keep in mind that these are the notes of an undergraduate student in the process of learning the material himself, so take what you read with a grain of salt. If you find mistakes and feel like telling me, I will be grateful and happy to hear from you, even for the most trivial of errors. You can reach me by email, in English, Slovene, or Spanish, at ejmastnak@gmail.com.

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Elastomechanics

A Few Vector Calculus Identities

$\phi : \mathbb{R}^3 \to \mathbb{R}$ denotes a well-behaved scalar field.
$\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes a well-behaved vector field.
$v \equiv |\mathbf{v}|$ (shorthand for vector magnitude)
$\nabla \times (\nabla \phi) = \mathbf{0}$ (curl of gradient is zero)
$\frac{1}{2}\nabla v^2 = \mathbf{v} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{v}$
$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$

Common Vector Operators By Components

$(\nabla \phi)_i = \frac{\partial \phi}{\partial x_i}$ (gradient of a scalar)
$(\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j}$ (gradient of a vector)
$\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i}$ (divergence of a vector)
$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_i^2}$ (Laplacian of a scalar)
$(\nabla^2 \mathbf{v})_i = \frac{\partial}{\partial x_j}\frac{\partial v_i}{\partial x_j}$ (Laplacian of a vector)
$\big[\nabla(\nabla \cdot \mathbf{v})\big]_i = \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}$ (gradient of divergence)
$\big[(\mathbf{v} \cdot \nabla)\mathbf{v}\big]_i = v_j \frac{\partial v_i}{\partial x_j}$ (convective derivative)

Geometry of Deformations

Displacement Vector

Consider a reference element in a continuous body at the position vector $\mathbf{x}$. Due to a deformation, the reference element shifts to the new position $\mathbf{x}'$.
$\mathbf{u} \equiv \mathbf{x}' - \mathbf{x}$ (displacement vector)
Lesson: $\mathbf{x}'$, and thus $\mathbf{u}$, are functions of the initial position $\mathbf{x}$.

Separation Between Neighboring Elements

Consider two neighboring reference elements initially connected by the position vector $\mathbf{x}$. After a deformation, the elements are connected by a new position vector $\mathbf{x}'$.
$(\mathrm{d}l)^2 = (\mathrm{d}x_i)^2$ (pre-deformation separation btwn. elements)
$\quad = (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 + (\mathrm{d}x_3)^2$ (written in full)
$\mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{x}' - \mathrm{d}\mathbf{x}$ (displacement vector)
$\mathrm{d}\mathbf{x}' = \mathrm{d}\mathbf{x} + \mathrm{d}\mathbf{u}$ (new position in terms of $\mathbf{u}$)
$(\mathrm{d}l')^2 = (\mathrm{d}x_i')^2$ (post-deformation separation btwn. elements)
$\quad = (\mathrm{d}x_i + \mathrm{d}u_i)^2$

Deriving the Strain Tensor

$\mathrm{d}u_i = \frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j$ ($\mathrm{d}u$ in terms of $\mathrm{d}x$)
Substitute this expression for $\mathrm{d}u_i$ into $(\mathrm{d}l')^2$ above to get...

$$(\mathrm{d}l')^2 = \left(\mathrm{d}x_i + \frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j\right)^2$$
$$= (\mathrm{d}x_i)^2 + 2\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j\,\mathrm{d}x_i + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\,\mathrm{d}x_j\,\mathrm{d}x_k$$

$(\mathrm{d}l')^2 - (\mathrm{d}l)^2 = 2\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j\,\mathrm{d}x_i + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\,\mathrm{d}x_j\,\mathrm{d}x_k$ (rearranged)
$2\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j\,\mathrm{d}x_i = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\mathrm{d}x_i\,\mathrm{d}x_j$ (symmetrization)
$\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\,\mathrm{d}x_j\,\mathrm{d}x_k = \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\,\mathrm{d}x_i\,\mathrm{d}x_j$ (changed dummy indices)
$\implies (\mathrm{d}l')^2 - (\mathrm{d}l)^2 = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)\mathrm{d}x_i\,\mathrm{d}x_j$

The above expression motivates the definition of the strain tensor as...
$u_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$ (**strain tensor**)
$(\mathrm{d}l')^2 - (\mathrm{d}l)^2 = 2u_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j$ (in terms of strain tensor)
$u_{ij}^{\mathrm{lin}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ (linear strain tensor)

Strain Tensor and Displacement Vector's Gradient

$u_{ij}^{\mathrm{lin}} = \frac{1}{2}\left[(\nabla \mathbf{u})_{ij} + (\nabla \mathbf{u})_{ji}\right]$ (linear ST in terms of $\nabla \mathbf{u}$)
$\mathsf{u}_{\mathrm{lin}} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right]$ (in vector notation)

$$(\nabla \mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{asymmetric}}$$

$\nabla \mathbf{u}$'s symmetric component is the linear strain tensor.
$\nabla \mathbf{u}$'s asymmetric component corresponds to rigid rotations.

Strain Tensor's Symmetry and Rigid Rotations

Strain tensor is made symmetric on physical grounds so that $u_{ij} = 0$ (no internal deformation) for rigid rotations.
Consider a rigid rotation about the z axis by $\delta\phi \ll 1$.

$$\mathbf{R} = \begin{pmatrix} \cos\delta\phi & -\sin\delta\phi & 0 \\ \sin\delta\phi & \cos\delta\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(rotation matrix)}$$

$$\mathbf{R} \approx \begin{pmatrix} 1 & -\delta\phi & 0 \\ \delta\phi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(for } \delta\phi \ll 1\text{)}$$

An initial position $\mathbf{x} = (x_1, x_2, x_3)^\top$ transforms as...

$$\mathbf{x}' = \mathbf{R}\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_2\delta\phi \\ x_1\delta\phi \\ 0 \end{pmatrix} \equiv \mathbf{x} + \mathbf{u}$$

Idea: by definition, rigid rotations don't deform bodies ($u_{ij} \equiv 0$)!
Because the strain tensor is symmetrized...
$u_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -\delta\phi + \delta\phi = 0$ (correctly, $u_{ij} = 0$)
$u_{21} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} = \delta\phi - \delta\phi = u_{12} = 0$ (correctly, $u_{ij} = 0$)
If the strain tensor were not symmetrized...
$\widetilde{u}_{12} = \frac{\partial u_1}{\partial x_2} = -\delta\phi$ (non-physically, $\widetilde{u}_{12} \neq 0$)
$\widetilde{u}_{21} = \frac{\partial u_2}{\partial x_1} = \delta\phi$ (non-physically, $\widetilde{u}_{12} \neq 0$)

Physical Meaning of the Diagonal Components

No summation implied over α!
Diagonal components $u_{\alpha\alpha}$ encode extensional strains along the α coordinate axes, e.g. u_{xx} is extensional strain along the x axis.
Consider two neighboring body elements with reference separation Δl. A deformation then separates the elements to...
$(\Delta l')^2 = (\Delta l)^2 + 2u_{ik}\Delta x_i \Delta x_k$
Let $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ denote the strain tensor's principal axes.
$\Delta l' = \sqrt{1 + 2u_{\alpha\alpha}}\Delta l$ (elements with ref. spacing $\Delta\mathbf{x} = \Delta l\,\hat{\mathbf{e}}_1$)
$\Delta l' \approx (1 + u_{\alpha\alpha})\Delta l$ (for small strains $u_{\alpha\alpha}$)
$u_{\alpha\alpha} = \frac{\Delta l' - \Delta l}{\Delta l}$ (diag. components are extensional strains)

Physical Meaning of the Off-Diagonal Components

Consider two pairs of nearby body elements with separations...
$\Delta\mathbf{x}_1 = \Delta x_1\,\hat{\mathbf{e}}_1$ and $\Delta\mathbf{x}_2 = \Delta x_2\,\hat{\mathbf{e}}_2$
A deformation then transforms the separations to...

$$\Delta\mathbf{x}_1' = \Delta\mathbf{x}_1 + \frac{\partial \mathbf{u}}{\partial x_1}\Delta x_1 = \left[\left(1 + \frac{\partial u_1}{\partial x_1}\right)\hat{\mathbf{e}}_1 + \frac{\partial u_2}{\partial x_1}\,\hat{\mathbf{e}}_2 + \frac{\partial u_3}{\partial x_1}\,\hat{\mathbf{e}}_3\right]\Delta x_1$$
$$\Delta\mathbf{x}_2' = \Delta\mathbf{x}_2 + \frac{\partial \mathbf{u}}{\partial x_2}\Delta x_2 = \left[\frac{\partial u_1}{\partial x_2}\,\hat{\mathbf{e}}_1 + \left(1 + \frac{\partial u_2}{\partial x_2}\right)\hat{\mathbf{e}}_2 + \frac{\partial u_3}{\partial x_2}\,\hat{\mathbf{e}}_3\right]\Delta x_2$$

To lowest order in products of Δx_1, Δx_2 and $\frac{\partial u_i}{\partial x_j}$...

$$\Delta\mathbf{x}_1' \cdot \Delta\mathbf{x}_2' \approx \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\Delta x_1 \Delta x_2 = 2u_{12}\Delta x_1 \Delta x_2.$$

$\cos\theta_{12} = \frac{\Delta\mathbf{x}_1' \cdot \Delta\mathbf{x}_2'}{|\Delta\mathbf{x}_1'||\Delta\mathbf{x}_2'|} \approx 2u_{12}$ (angle between $\Delta\mathbf{x}_1'$ and $\Delta\mathbf{x}_2'$)
$\theta_{12} \approx \frac{\pi}{2} - 2u_{12}$ (using $\arccos x = \frac{\pi}{2} - x + \mathcal{O}(x^3)$)
$\Delta\theta_{12} \equiv \frac{\pi}{2} - \theta_{12} = 2u_{12}$
$u_{\alpha\beta} = \frac{1}{2}\Delta\theta_{\alpha\beta}$ (meaning of off-diagonal components)
$\Delta\theta_{\alpha\beta}$ is the post-deformation reduction in angle (from the initially perpendicular value $\pi/2$) between a pair of line elements $\Delta\mathbf{x}_\alpha$ and $\Delta\mathbf{x}_\beta$ initially parallel to $\hat{\mathbf{e}}_\alpha$ and $\hat{\mathbf{e}}_\beta$, respectively.

Relative Change in Volume

Consider a cuboid reference body element with volume ΔV.
A deformation transforms the element to have volume $\Delta V'$.
$x_i' = x_i + u_i(x_j)$ (post-deformation coordinates)
$\mathbf{J}_{ij} \equiv \frac{\partial x_i'}{\partial x_j} = \frac{\partial x_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$ (Jacobian matrix)

$$\mathbf{J} = \begin{pmatrix} 1 + \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1 + \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1 + \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

$\Delta V' \approx \det \mathbf{J} \cdot \Delta V$ (to lowest order in Δx_j)
$\det \mathbf{J} \approx 1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 1 + \operatorname{tr} \mathsf{u}$ (to first order in $\frac{\partial u_i}{\partial x_j}$)
$\operatorname{tr} \mathsf{u} = \frac{\Delta V' - \Delta V}{\Delta V}$ ($\operatorname{tr} \mathsf{u}$ gives relative change in volume)

Shear and Isotropic Decomposition
Goal: decompose strain tensor into two parts representing...
(i) a *shear* deformation, which is a change in shape (i.e. the body's proportions) that preserves the body's volume, and
(ii) an *isotropic* deformation, which is a change in volume that preserves the body's shape.
$$u_{ij} = \underbrace{\left(u_{ij} - \frac{1}{3}u_{kk}\delta_{ij}\right)}_{\text{shear}} + \underbrace{\frac{1}{3}u_{kk}\delta_{ij}}_{\text{isotropic}} \qquad \text{(decomposition)}$$
$\operatorname{tr} \mathsf{u}_{\text{shear}} = 0 \implies$ no relative change in volume
$\mathsf{u}_{\text{isotropic}}$ is isotropic by construction (because of δ_{ij})

Lagrange Strain Tensor
The discussion so far has implicitly used the *Lagrange form* of the strain tensor. In the Lagrange approach, deformations are given in a coordinate system determined by pre-deformation reference state.
$\mathbf{x}'(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$ (in terms of $\mathbf{x}$)
$u_{ij}^{\text{L}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$ (Lagrange strain tensor)

The Euler Strain Tensor
Euler approach: deformations are given in a coordinate system determined by post-deformation state.
$\mathbf{x}' = \mathbf{x}(\mathbf{x}') - \mathbf{u}(\mathbf{x}')$ (in terms of $\mathbf{x}'$)
$\Delta x_i = \Delta x_i' - \frac{\partial u_i}{\partial x_j'}\Delta x_j'$
$(\Delta l')^2 - (\Delta l)^2 = \left(\frac{\partial u_i}{\partial x_k'} + \frac{\partial u_k}{\partial x_i'} - \frac{\partial u_l}{\partial x_i'}\frac{\partial u_l}{\partial x_k'}\right)\Delta x_i' \Delta x_k'$
$(\Delta l')^2 - (\Delta l)^2 \equiv 2u_{ik}^{\text{E}}\Delta x_i' \Delta x_k'$
$u_{ij}^{\text{E}} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j'} + \frac{\partial u_j}{\partial x_i'} - \frac{\partial u_k}{\partial x_i'}\frac{\partial u_k}{\partial x_j'}\right)$ (Euler strain tensor)

Mechanical Stress

An undeformed body is in a state of thermodynamic equilibrium.
$\implies$ the net force on any body element is zero.
Deformation rearranges the body's molecules, giving rise to forces tending to return the body to equilibrium. These forces are called *internal stress.*
Assumption: internal stresses are of short range (e.g. molecular scale). Thus, neighboring body elements interact with each other only via their mutually contacting surfaces.

The Stress Tensor
Consider a small body element with volume $\mathrm{d}V$ at position $\mathbf{r}$.
$\mathrm{d}\mathbf{F}_{\text{tot}} = \mathbf{f}_{\text{tot}}(\mathbf{r})\,\mathrm{d}V$ (total force on body element)
$\mathbf{f}_{\text{tot}} = \mathbf{f} + \mathbf{f}_{\text{ext}}$ (force decomposition)
$\mathbf{f}$ is density of short-range internal stress forces
$\mathbf{f}_{\text{ext}} = \rho\mathbf{a}_{\text{ext}}$ is density of external forces (e.g. weight)
$F_i = \iiint f_i\,\mathrm{d}V + \iiint f_i^{\text{ext}}\,\mathrm{d}V$ (total force on body)
Under short-range force assumption, internal stresses may be written as a *surface integral* (instead of a volume integral).
Idea: by the divergence theorem for tensors, a VI of a rank n tensor may be transformed to a SI of a rank $n+1$ tensor if the VI's integrand is the divergence of a rank $n+1$ tensor.
$\implies$ f_i must be the divergence of a rank two tensor, i.e.
$f_i = \frac{\partial p_{ij}}{\partial x_j}$ (implicit definition of **stress tensor**)
$\iiint f_\text{i}\,\mathrm{d}V \equiv \oiint p_{ij}\,\mathrm{d}S_j$ (internal stress as surface integral)

Torque and the Stress Tensor's Symmetry
Under assumption of short-range, surface contact forces between neighboring body elements, stress-induced torque must be expressible as a surface integral (like force was above).
$M_i = \iiint (\mathbf{r}\times\mathbf{f})_i\,\mathrm{d}V$ (torque on a body from internal stress)
$M_i = \iiint \epsilon_{ijk}x_j f_k\,\mathrm{d}V = \iiint \epsilon_{ijk}x_j \frac{\partial p_{kl}}{\partial x_l}\,\mathrm{d}V$
$\quad = \iiint \epsilon_{ijk}\frac{\mathrm{d}[x_j p_{kl}]}{\mathrm{d}x_l}\,\mathrm{d}V - \iiint \epsilon_{ijk}p_{kl}\frac{\partial x_j}{\partial x_l}\,\mathrm{d}V$
$\quad = \oiint \epsilon_{ijk}x_j p_{kl}\,\mathrm{d}S_l - \iiint \epsilon_{ijk}p_{kl}\delta_{jl}\,\mathrm{d}V$
Idea: stress-induced torque is expressible as a surface integral if the above volume integral vanishes.
$\epsilon_{ijk}\delta_{jl}p_{kl} = \epsilon_{ijk}p_{kj} = \frac{1}{2}\left(\epsilon_{ijk}p_{kj} + \epsilon_{ikj}p_{jk}\right) = \frac{\epsilon_{ijk}}{2}\left(p_{kj} - p_{jk}\right)$
$\epsilon_{ijk}\delta_{jl}p_{kl} = 0$ (and $\mathbf{M}_\text{i}$ becomes a surface integral) if $p_{kj} = p_{jk}$

Stress Tensors for Common Deformations
$\alpha, \beta \in \{x, y, z\}$
No summation is implied over α or β!
$p_{ij} = -p\delta_{ij};\ p > 0$ (isotropic compression)
$$\mathsf{p} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \qquad \text{(isotropic compression, matrix form)}$$
$p_{ij} = p(\delta_{i\alpha}\delta_{k\beta} + \delta_{i\beta}\delta_{k\alpha});\ \alpha \neq \beta$ (shear deformation)
$$\mathsf{p} = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \\ p & 0 & 0 \end{pmatrix} \qquad \text{(shear deformation with } \alpha = x,\ \beta = z\text{)}$$
$p_{ij} = p\delta_{i\alpha}\delta_{k\alpha}$ (uniaxial load)
$$\mathsf{p} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix} \qquad \text{(uniaxial load with } \alpha = z\text{)}$$

The Cauchy Equation
$\mathrm{d}F_\text{i}^{\text{tot}} = f_\text{i}^{\text{tot}}(\mathbf{r})\,\mathrm{d}V = \rho\ddot{x}_i\,\mathrm{d}V$ (Newton's law in general)
$\mathrm{d}F_i^{\text{tot}} = f_i^{\text{tot}}(\mathbf{r})\,\mathrm{d}V = \left(\frac{\partial p_{ij}}{\partial x_j} + f_i^{\text{ext}}\right)\mathrm{d}V$ (total force on a body element $\mathrm{d}V$ in the presence of stress $\frac{\partial p_{ij}}{\partial x_j}$)
Comparison to Newton's law, letting $\ddot{x}_i \to \ddot{u}_i$, produces...
$\rho\ddot{u}_i = \frac{\partial p_{ij}}{\partial x_j} + f_i^{\text{ext}}$ (**Cauchy equation**)

Work During a Deformation
Consider the work δW done by internal stress forces during a small deformation with displacement vector δu_i.
$\delta W = \iiint_V f_i\delta u_i\,\mathrm{d}V = \iiint_V \frac{\partial p_{ij}}{\partial x_j}\delta u_i\,\mathrm{d}V$
$\quad = \oiint_{\partial V} p_{ij}\delta u_i\,\mathrm{d}S_j - \iiint_V p_{ij}\delta u_{ij}\,\mathrm{d}V$
$\quad$ (using $\frac{\partial p_{ij}}{\partial x_j}\delta u_i = \frac{\partial[p_{ij}\delta u_i]}{\partial x_j} - p_{ij}\frac{\partial\delta u_i}{\partial x_j}$)
Neglect stress on the body element's surface and get...
$\delta W = -\iiint_V p_{ij}\delta u_{ij}\,\mathrm{d}V$ (neglecting surface stresses)
$\delta w = -p_{ij}\delta u_{ij}$ (normalized by volume)
$\mathrm{d}F = -S\,\mathrm{d}T - \delta W$ (from thermodynamics)
$\mathrm{d}f = -s\,\mathrm{d}T - \delta w$ (normalized by volume)
(F is free energy density; S is entropy; W is work)
$\mathrm{d}f = -s\,\mathrm{d}T + p_{ij}\,\mathrm{d}u_{ij}$
$p_{ij} = \left(\frac{\partial f}{\partial u_{ij}}\right)_T$ (**stress in terms of free energy**)

Elastic Free Energy

Restriction: consider only isotropic materials.
Restriction: consider only isothermal deformations ($\mathrm{d}T = 0$).
Isotropic bodies are invariant under rotations, translations, reflections... $\implies$ a body's free energy can depend only on quantities that are also invariant under these transformations.

Elastic Free Energy and Hooke's Law
A body's free energy f depends on the strain tensor u.
f can depend only on u's *invariant quantities.*
As invariant quantities, we take u's eigenvalues.
$\det(\mathsf{u} - \lambda\mathsf{I}) =$ (u's characteristic polynomial)
$\qquad = -\lambda^3 + (\operatorname{tr}\mathsf{u})\lambda^2 - \frac{1}{2}\left[(\operatorname{tr}\mathsf{u})^2 - \operatorname{tr}u^2\right]\lambda + \det\mathsf{u}$
Eigenvalues are fully determined by char. poly. coefficients
$\implies$ we can use only the char. poly. coefficients as invariants
$\operatorname{tr}\mathsf{u}, \quad \operatorname{tr}\mathsf{u}^2, \quad \det\mathsf{u}$ (inv. quantities used to determine f)
Expand f in invariant quantities (rejecting mixed terms):

$f = f_0 + A_1 \operatorname{tr} \mathsf{u} + A_2 (\operatorname{tr} \mathsf{u})^2 + \cdots$
$\quad + B_1 \operatorname{tr} \mathsf{u}^2 + (\operatorname{tr} \mathsf{u}^2)^2 + \cdots$
$\quad + C_1 \det \mathsf{u} + (\det \mathsf{u})^2 + \cdots$

Restriction: restrict analysis to linear theory, in which strain is linear in stress and f is quadratic in displacement u.

Resulting simplification: reject the linear term $A_1 \operatorname{tr} \mathsf{u}$.

$f \approx f_0 + A_2 (\operatorname{tr} \mathsf{u})^2 + B_1 \operatorname{tr} \mathsf{u}^2$ (assuming linear theory)
$= f_0 + A_2 (u_{kk})^2 + B_1 u_{ij} u_{ji}$ (by components)
$\equiv f_0 + \frac{\lambda}{2}(u_{kk})^2 + \mu u_{ij} u_{ji}$ (define **Lamé constants** λ and μ)

$p_{ij} = \left(\frac{\partial f}{\partial u_{ij}}\right)_T = \lambda u_{kk} \delta_{ij} + 2\mu u_{ij}$

$p_{ij} = \lambda u_{kk} \delta_{ij} + 2\mu u_{ij}$ (**Hooke's law**)

Interpretation: stress (p_{ij}) is proportional to the sum of diagonal strain ($u_{kk}\delta_{ij}$, prop. constant λ) and shear strain ($2u_{ij}$, prop. constant μ).

Naively: remember factor of two from $u_{ij} = u_{ji}$.

$u_{kk} = \frac{p_{kk}}{3\lambda + 2\mu}$ (diagonal stress and strain)

$u_{ij} = \frac{1}{2\mu}\left(p_{ij} - \frac{\lambda p_{kk}}{3\lambda + 2\mu}\delta_{ij}\right)$ (**constitutive relation**)

The Bulk and Shear Moduli

First decompose u_{ik} into shear and isotropic components...

$u_{ij} = u_{ij}^{\text{shear}} + u_{ij}^{\text{isotropic}} = \left(u_{ij} - \frac{1}{3}u_{kk}\delta_{ij}\right) + \frac{1}{3}u_{kk}\delta_{ij}$

Then compute free energy in terms of components...

$f \equiv f_0 + \widetilde{\mu} u_{ij}^{\text{sh}} u_{ji}^{\text{sh}} + \frac{3K}{2} u_{ij}^{\text{iso}} u_{ji}^{\text{iso}}$ ($\widetilde{\mu}, K$ to be determined)
$= f_0 + \widetilde{\mu}\left(u_{ij} - \frac{1}{3}u_{kk}\delta_{ij}\right)^2 + \frac{1}{2}K u_{kk}^2$
$= f_0 + \frac{1}{2}\left(K - \frac{2\widetilde{\mu}}{3}\right)u_{kk}^2 + \widetilde{\mu} u_{ij} u_{ji}$

...then match terms with $f = f_0 + \frac{\lambda}{2}u_{kk}^2 + \mu u_{ij} u_{ji}$ to get...

$\implies \widetilde{\mu} = \mu \qquad K = \lambda + \frac{2\mu}{3}$

K is *bulk modulus* (resistance to strain from isotropic stress)

μ is *shear modulus* (resistance to strain from shear stress)

Bulk Modulus

Consider an isotropic compression $p_{ij} = -p\delta_{ij}$.

$u_{kk} = \operatorname{tr} \mathsf{u} = \frac{\Delta V}{V}$ (u_{kk} is relative volume change)

$\frac{\mathrm{d}V}{V} = \alpha \,\mathrm{d}T - \frac{\mathrm{d}p}{K}$ (from thermodynamics)

$\frac{\mathrm{d}V}{V} = -\frac{\mathrm{d}p}{K}$ (if $\mathrm{d}T = 0$)

$\frac{\Delta V}{V} = u_{kk} = -\frac{p}{\lambda + 2\mu/3}$ (from constitutive relation)

Compare to $\frac{\mathrm{d}V}{V} = -\frac{\mathrm{d}p}{K} \implies K = \lambda + 2\mu/3$

Lesson: $\lambda + 2\mu/3$ is consistent with bulk modulus K from TD.

Young's Modulus and the Poisson Ratio

Consider a uniaxial load along the γ axis.

$p_{ij} = p\delta_{i\gamma}\delta_{j\gamma}$ (uniaxial stress tensor; no summation over γ)

$u_{\alpha\alpha} = u_{\beta\beta} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}p$ (from const. relation)

$u_{\gamma\gamma} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}p$ (from const. relation)

$E \equiv \frac{p}{u_{\gamma\gamma}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ (definition of Young's modulus)

$\nu \equiv -\frac{u_{\alpha\alpha}}{u_{\gamma\gamma}} = -\frac{u_{\beta\beta}}{u_{\gamma\gamma}} = \frac{\lambda}{2(\lambda + \mu)}$ (definition of Poisson's ratio)

E encodes resistance to axial strain under uniaxial stress.

ν is ratio of transverse and axial strain under uniaxial stress.

$\nu \in (-1, 1/2)$ (physically permissible bounds)

$\nu \in (0, 1/2)$ (for typical materials)

Relationship Between Elastic Constants

$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ (E in terms of Lamé constants)

$\nu = \frac{\lambda}{2(\lambda + \mu)}$ (ν in terms of Lamé constants)

$\lambda = \frac{\nu E}{(1 - 2\nu)(1 + \nu)}$ (Lamé constants in terms of ν and E)

$\mu = \frac{E}{2(1 + \nu)}$ (Lamé constants in terms of ν and E)

$K = \frac{E}{3(1 - 2\nu)}$ (bulk modulus in terms of ν and E)

Important Relationship In Terms of E and ν

$f = f_0 + \frac{E}{2}\left[\frac{\nu}{(1 + \nu)(1 - 2\nu)}u_{kk}^2 + \frac{1}{1 + \nu}u_{ij}u_{ji}\right]$ (**free energy**)

$p_{ij} = \frac{E}{(1 + \nu)}\left(u_{ij} + \frac{\nu}{1 - 2\nu}u_{kk}\delta_{ij}\right)$ (**Hooke's law**)

$u_{ij} = \frac{1}{E}\left[(1 + \nu)p_{ij} - \nu p_{kk}\delta_{ij}\right]$ (**constitutive relation**)

Navier Equation

The Navier equation describes the dynamics of linear, isotropic bodies in terms of the displacement vector $\mathbf{u}$.

$\rho \ddot{u}_i = \frac{\partial p_{ij}}{\partial x_j} + f_i^{\text{ext}}$ (Cauchy equation, for reference)

$p_{ij} = \lambda u_{kk}\delta_{ij} + 2\mu u_{ij}$ (Hooke's law, for reference)

$\frac{\partial p_{ij}}{\partial x_j} = \lambda \frac{\partial u_{jj}}{\partial x_i} + 2\mu \frac{\partial u_{ij}}{\partial x_j}$ (using Hooke's law)
$= \lambda \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \mu \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}$ (using strain tensor)
$= (\lambda + \mu)\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$ (factoring common terms)
$= (\lambda + \mu)\left[\nabla\nabla \cdot \mathbf{u}\right]_i + \mu(\nabla^2 \mathbf{u})_i$ (vector form)

$\rho \ddot{\mathbf{u}} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \lambda \nabla^2 \mathbf{u} + \mathbf{f}_{\text{ext}}$ (**Navier equation**)

$\rho \ddot{\mathbf{u}} = \frac{E}{2(1 + \nu)}\left(\nabla^2 \mathbf{u} + \frac{1}{1 - 2\nu}\nabla(\nabla \cdot \mathbf{u})\right) + \mathbf{f}_{\text{ext}}$ (using E and ν)

Plates

A plate is a material whose thickness is much smaller than its surface dimensions.

Consider a plate of thickness h whose neutral plane aligns with the xy plane in the undeformed reference state.

Let $\zeta(x, y)$ denote plate displacement along z axis of a point in the deformed neutral plane from the xy plane.

Assumption: restrict analysis to small flexure ($\zeta \ll h$), for which deformation in transverse dimension dominates deformation in longitudinal dimensions.

$u_x \approx u_y \approx 0$ (displacement vector assuming small flexure)

$u_z \approx \zeta(x, y)$ (displacement vector assuming small flexure)

Flexure for Thin Plates

Concept: for a thin plate, even small surface forces cause considerable flexure $\implies$ internal stresses from deformation are much larger than external surface forces that cause them.

Let $\hat{\mathbf{n}}_\pm$ denote normals to upper and lower plate surfaces.

$\hat{\mathbf{n}}_+ \approx (0, 0, +1)$ (assuming small flexure)

$\hat{\mathbf{n}}_- \approx (0, 0, -1)$ (assuming small flexure)

Let $\mathbf{f}$ denote density of stress forces per unit plate area

$f_i\big|_{\text{surface}} = p_{ij}n_j\big|_{\text{surface}} \approx 0$ (for negligible surface forces)

$\left[p_{xz} \approx p_{yz} \approx p_{zz} \approx 0\right]_{\text{surface}}$ (assuming $p_{ij}n_j\big|_{\text{surface}} \approx 0$)

Concept: for a thin plate, since p must change continuously, if components $p_{ij} \approx 0$ on both plate surfaces, then $p_{ij} \approx 0$ throughout the plate volume. Resultantly....

$\left[p_{xz} \approx p_{yz} \approx p_{zz} \approx 0\right]_{\text{throughout plate}}$ (assuming thin plate)

Strain Tensor of a Flexed Plate

$p_{ij} = \lambda u_{kk}\delta_{ij} + 2\mu u_{ij}$ (Hooke's law in general)
$= \frac{E}{1 + \nu}\left(\frac{\nu}{1 - 2\nu}u_{kk}\delta_{ij} + u_{ij}\right)$ (in terms of ν and E)

$p_{xz} \approx p_{yz} \approx p_{zz} \approx 0$ (for a thin plate and small flexure)

$u_{xz} \approx u_{yz} \approx 0$ (from Hooke's law and $p_{xz} \approx p_{yz} \approx 0$)

$u_{zz} \approx \frac{\nu}{\nu - 1}(u_{xx} + u_{zz})$ (from Hooke's law and $p_{zz} \approx 0$)

$\frac{\partial u_x}{\partial z} = -\frac{\partial u_z}{\partial x} = -\frac{\partial \zeta(x, y)}{\partial x}$ (using $u_{xz} \approx 0$ and $u_z \approx \zeta$)

$\implies u_x = -z\frac{\partial \zeta}{\partial x}$ (assuming $u_x = 0$ at $z = 0$)

$u_{xx} = \frac{\partial u_x}{\partial x} = -z\frac{\partial^2 \zeta}{\partial x^2}$

$\frac{\partial u_y}{\partial z} = -\frac{\partial u_z}{\partial y} = -\frac{\partial \zeta(x, y)}{\partial y}$ (using $u_{yz} \approx 0$ and $u_z \approx \zeta$)

$\implies u_y = -z\frac{\partial \zeta}{\partial y}$ (assuming $u_y = 0$ at $z = 0$)

$u_{yy} = \frac{\partial u_y}{\partial y} = -z\frac{\partial^2 \zeta}{\partial y^2}$

$u_{zz} \approx \frac{\nu}{\nu - 1}(u_{xx} + u_{yy})$ (from above)
$\approx \frac{\nu z}{1 - \nu}\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right)$ (using above u_{xx} and u_{yy})

$$\mathsf{u} = \begin{pmatrix} -z\frac{\partial^2\zeta}{\partial x^2} & -z\frac{\partial^2\zeta}{\partial x\partial y} & 0 \\ -z\frac{\partial^2\zeta}{\partial x\partial y} & -z\frac{\partial^2\zeta}{\partial y^2} & 0 \\ 0 & 0 & \frac{\nu z}{1-\nu}\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)\end{pmatrix} \qquad \text{(thin plate)}$$

Free Energy Density of a Thin Plate

$$\mathsf{u} \equiv \begin{pmatrix} u_{xx} & u_{xy} & 0 \\ u_{xy} & u_{yy} & 0 \\ 0 & 0 & \frac{\nu}{\nu-1}(u_{xx}+u_{yy})\end{pmatrix} \qquad \text{(thin plate, shorthand)}$$

$u_{xx} = -z\frac{\partial^2\zeta}{\partial x^2}$

$u_{yy} = -z\frac{\partial^2\zeta}{\partial y^2}$

$u_{xy}^2 = -z\frac{\partial^2\zeta}{\partial x\partial y}$

$\operatorname{tr}\mathsf{u} = u_{kk} = \frac{1-2\nu}{1-\nu}(u_{xx}+u_{yy})$ (trace of thin plate u)

$$\mathsf{u}^2 = \begin{pmatrix} u_{xx}^2+u_{xy}^2 & - & - \\ - & u_{xy}^2+u_{yy}^2 & - \\ - & - & \left(\frac{\nu}{1-\nu}\right)^2(u_{xx}+u_{yy})^2\end{pmatrix}$$

$$\begin{aligned}\operatorname{tr}\mathsf{u}^2 = u_{ij}u_{ji} &= u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 + \left(\frac{\nu}{1-\nu}\right)^2(u_{xx}+u_{yy})^2 \\ &= (u_{xx}+u_{yy})^2 + \left(\frac{\nu}{1-\nu}\right)^2(u_{xx}+u_{yy})^2 + 2u_{xy}^2 - 2u_{xx}u_{yy} \\ &= \frac{1-2\nu+2\nu^2}{(1-\nu)^2}(u_{xx}+u_{yy})^2 + 2u_{xy}^2 - 2u_{xx}u_{yy}\end{aligned}$$

$f = \frac{E}{2(1+\nu)}\left[\frac{\nu}{(1-2\nu)}u_{kk}^2 + u_{ij}u_{ji}\right]$ (in general)

Then substitute in above u_{kk} and $u_{ij}u_{ji}$ to get...

$f = \frac{E}{2(1+\nu)(1-\nu)}\left[(u_{xx}+u_{yy})^2 + 2(1-\nu)(u_{xy}^2 - u_{xx}u_{yy})\right]$

Finally substitute in expressions for u_{xx}, u_{xx}, u_{xy} to get...

$$f = \frac{Ez^2}{2(1-\nu^2)}\left\{\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)^2 + 2(1-\nu)\left[\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2 - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right]\right\}$$

Conclusion: f grows as $f\sim z^2$ for transverse displacements.

Free Energy of a Thin Plate

Consider a plate of thickness h.

$$\begin{aligned}F &= \iiint f\,\mathrm{d}^3\mathbf{r} \qquad \text{(general expression for free energy)} \\ &= F_0 + \int_{-h/2}^{h/2}\frac{Ez^2}{2(1-\nu^2)}\,\mathrm{d}z\int\left\{\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)^2\right. \qquad \text{(using above } f\text{)} \\ &\qquad \left.+2(1-\nu)\left[\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2 - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right]\right\}\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

$$\begin{aligned}F = F_0 + \frac{Eh^3}{24(1-\nu^2)}\iint&\left\{\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)^2\right. \qquad \text{(integrating over } z\text{)} \\ &\left.+2(1-\nu)\left[\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2 - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right]\right\}\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

$D \equiv \frac{Eh^3}{12(1-\nu^2)}$ (the plate's *flexural rigidity*)

$$\begin{aligned}F = F_0 + \frac{D}{2}\iint&\left\{\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)^2\right. \qquad \text{(plate's } F \text{ in terms of } D\text{)} \\ &\left.+2(1-\nu)\left[\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2 - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right]\right\}\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Flexure and Equilibrium

Idea: analyze how a external displacing force and plate free energy change during a *virtual displacement* in equilibrium.

$\delta F = \iint P\delta\zeta\,\mathrm{d}S$ (variational condition for plate equilibrium)

P is surface density of external forces on plate.

δF is variation in the plate's elastic free energy.

$\delta\zeta$ is variation in transverse displacement ζ.

Let $\Delta \equiv \nabla^2$. (alternate notation for Laplacian)

Without proof, we quote the following variational result:

$\iint(D\Delta^2\zeta - P)\delta\zeta\,\mathrm{d}S = 0$ (without derivation)

$\implies D\Delta^2\zeta = P$ (**equilibrium condition**)

Simple Plate Boundary Conditions

Let $\hat{\mathbf{m}}$ be outward normal to plate's circumference (and *not* the normal to the plate surface).

Let $\hat{\mathbf{l}}$ be tangent to plate's circumference.

$\zeta(x,y)\big|_{\text{edge}} = 0$ (for plates with fixed or supported edges)

$\frac{\partial\zeta}{\partial\hat{\mathbf{m}}}\Big|_{\text{edge}} = 0$ (for plates with fixed edges)

$\left[\frac{\partial^2\zeta}{\partial m^2} + \nu\frac{\partial\theta}{\partial l}\frac{\partial\zeta}{\partial m}\right]_{\text{edge}} = 0$ (supported edges)

Longitudinal Loads and Buckling

In a longitudinal load, loading forces act in the plane of plate.

Goal: determine critical load at which plate buckles.

Limitation: the linear theory used here can predict critical load, but not the shape of the resulting deformation.

$\delta W = -\iiint p_{ik}\delta u_{ik}\,\mathrm{d}V$ (virtual work in 3D)

$\delta W = -\iint\widetilde{p}_{ik}\delta\widetilde{u}_{ik}\,\mathrm{d}S$ (virtual work in 2D)

$\widetilde{p}$ and $\widetilde{u}$ denote 2D stress and strain tensors.

Assumption: assume stress is known and constant.

$W = -\iint\widetilde{p}_{ik}\widetilde{u}_{ik}\,\mathrm{d}S$ (total work for fixed stress)

TODO: Critical Buckling Load

Rods

Rods are objects with longitudinal dimensions much larger than their cross-sectional dimensions.

Rod Torsion

Torsion: fix one end of rod and rotate the opposite end relative to the fixed end along the rod's longitudinal axis.

a is rod's linear cross-sectional dimension.

z is the rod's longitudinal coordinate.

ϕ is angle of rotation during torsion.

$\boldsymbol{\phi} = \phi\,\hat{\mathbf{e}}_z$ is corresponding vector torsion angle.

$\mathrm{d}\phi = \tau\,\mathrm{d}z$ (definition of torsion constant τ)

τ is torsion angle per unit longitudinal length along rod.

Restriction: we consider only small torsion with $\tau a \ll 1$.

$\phi \approx \tau z$ (for $\tau a \ll 1$)

$\mathbf{u}(\mathbf{r}) = \boldsymbol{\phi}\times\mathbf{r} + \mathcal{O}(\boldsymbol{\phi}^2)$ (torsion displacement vector)

$\mathbf{u} = (\tau z\,\hat{\mathbf{e}}_z)\times\mathbf{r} = \tau z(x\,\hat{\mathbf{e}}_y - y\,\hat{\mathbf{e}}_x)$

Limitation: above expression for $\mathbf{u}$, which holds only to lowest order in $\boldsymbol{\phi}$, predicts $u_z = 0$, i.e. that rod length does not change during torsion. From experiment, $u_z \neq 0$ during torsion, although $u_z \neq u_z(z)$. To match experiment, we define...

$u_z = \tau\psi(x,y)$ (ansatz for u_z; ψ is *torsion function*)

Strain Tensor for Torsion

$\mathbf{u} = \tau\big(-yz, xz, \psi(x,y)\big)$ (torsion displacement vector)

Restriction: consider only linear strain tensor.

$u_{ij}^{\text{lin}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ (linear strain tensor)

$u_{xz} = \frac{\tau}{2}\left(-y + \frac{\partial\psi}{\partial x}\right)$

$u_{yz} = \frac{\tau}{2}\left(x + \frac{\partial\psi}{\partial y}\right)$

$u_{xx} = u_{yy} = u_{zz} = u_{xy} = u_{yx} = 0$ (from above $\mathbf{u}$)

$\implies \operatorname{tr}\mathsf{u} = 0$ (torsion is a shear deformation)

$$\mathsf{u} = \begin{pmatrix} 0 & 0 & u_{xz} \\ 0 & 0 & u_{yz} \\ u_{xz} & u_{yz} & 0\end{pmatrix} \qquad \text{(torsion strain tensor)}$$

Stress Tensor for Torsion

$p_{ij} = \lambda u_{kk}\delta_{ij} + 2\mu u_{ij}$ (Hooke's law in general)

$p_{ij} = 2\mu u_{ij}$ (for torsion with $u_{kk} = 0$)

$p_{xz} = \tau\mu\left(-y + \frac{\partial\psi}{\partial x}\right)$

$p_{yz} = \tau\mu\left(x + \frac{\partial\psi}{\partial y}\right)$

$p_{xx} = p_{yy} = p_{zz} = p_{xy} = p_{yx} = 0$ (from above u)

$$\mathsf{p} = \begin{pmatrix} 0 & 0 & p_{xz} \\ 0 & 0 & p_{yz} \\ p_{xz} & p_{yz} & 0\end{pmatrix} \qquad \text{(torsion stress tensor)}$$

Force Density Components for Torsion

$f_x = \frac{\partial p_{xj}}{\partial x_j} = \mu\tau \frac{\partial}{\partial z}\left(-y + \frac{\partial \psi(x,y)}{\partial x}\right) = 0$

$f_y = \frac{\partial p_{yj}}{\partial x_j} = \mu\tau \frac{\partial}{\partial z}\left(x + \frac{\partial \psi(x,y)}{\partial y}\right) = 0$

$f_z = \frac{\partial p_{zj}}{\partial x_j} = \mu\tau \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$

$\equiv \mu\tau \nabla^2_{xy}\psi$

$\nabla^2_{xy}\psi = \frac{f_z}{\mu\tau}$ (in general: Poisson equation)

In equilibrium, internal stress force is zero throughout the rod.

$f_i = \frac{\partial p_{ij}}{\partial x_j} = 0$ (equilibrium condition)

$\nabla^2_{xy}\psi = 0$ (in equilibrium: Laplace equation)

Associated Torsion Function

Motivation: determine a new torsion function χ which simplifies boundary conditions on the equilibrium Laplace equation.

$p_{xz} = 2\mu\tau\frac{\partial \chi}{\partial y}$ (implicit definition of χ)

$p_{yz} = -2\mu\tau\frac{\partial \chi}{\partial x}$ (implicit definition of χ)

$\frac{\partial \psi}{\partial x} = 2\frac{\partial \chi}{\partial y} + y$ (relating χ and ψ)

$\frac{\partial \psi}{\partial y} = -2\frac{\partial \chi}{\partial x} - x$ (relating χ and ψ)

$\nabla^2_{xy}\chi = -1$ (after differentiating and rearranging)

Associated Torsion Function and Boundary Conditions

$\mathbf{l} = (\mathrm{d}x, \mathrm{d}y, 0)$ is tangent to rod's cross sectional perimeter

$\hat{\mathbf{n}}$ is normal to rod's lateral surface.

$\hat{\mathbf{n}} = \frac{1}{\mathrm{d}l}(-\mathrm{d}y, \mathrm{d}x, 0)$ ($\hat{\mathbf{n}}$ in terms of $\hat{\mathbf{l}}$)

$f_i\big|_{\text{lat}} = p_{ij}n_j\big|_{\text{lat}} = 0$ (rod is unloaded on lateral surface)

$f_z = p_{zj}n_j = -2\mu\tau\left(\frac{\mathrm{d}x}{\mathrm{d}l}\frac{\partial \chi}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}l}\frac{\partial \chi}{\partial y}\right)$ (in general)

$f_z = 0 \implies \frac{\mathrm{d}x}{\mathrm{d}l}\frac{\partial \chi}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}l}\frac{\partial \chi}{\partial y} = 0$ (at rod surface)

$\frac{\partial \chi}{\partial x}\,\mathrm{d}x + \frac{\partial \chi}{\partial y}\,\mathrm{d}y = \mathrm{d}\chi = 0$ (after canceling $1/\mathrm{d}l$)

$\mathrm{d}\chi\big|_{\text{surf}} = 0$ (boundary condition on rod surface)

$\implies \chi\big|_{\text{surf}} = \text{constant}$ (boundary condition on rod surface)

Elastic Energy for Torsion

$p_{ij} = \lambda u_{kk}\delta_{ij} + 2\mu u_{ij}$ (Hooke's law)

$f = \frac{\lambda}{2}u_{kk}^2 + \mu u_{ij}^2$ (elastic free energy)

$u_{kk} = 0$ for torsion (from Strain Tensor for Torsion)

$p_{kk} = 0$ for torsion (from Stress Tensor for Torsion)

$p_{ij} = 2\mu u_{ij}$ (Hooke's law for torsion)

$f = \mu u_{ij}^2$ (free energy for torsion)

$= \frac{1}{2}u_{ij}p_{ij}$ (using torsion Hooke's law)

$= u_{xz}p_{xz} + u_{yz}p_{yz}$ (most p_{ij} and u_{ij} are zero)

$= \frac{1}{2\mu}(p_{xz}^2 + p_{yz}^2)$ (using torsion Hooke's law)

$p_{xz} = 2\mu\tau\frac{\partial \chi}{\partial y}$ (from Associated Torsion Function)

$p_{yz} = -2\mu\tau\frac{\partial \chi}{\partial x}$ (from Associated Torsion Function)

$f = 2\mu\tau^2\left[\left(\frac{\partial \chi}{\partial x}\right)^2 + \left(\frac{\partial \chi}{\partial y}\right)^2\right]$ (in terms of χ)

$= 2\mu\tau^2\left(\nabla\chi\cdot\nabla\chi\right)$

$F = \iiint f\,\mathrm{d}V = 2\mu\iiint \tau^2\left(\nabla\chi\right)^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

Let $\mathcal{T} \equiv 4\mu\iint(\nabla\chi)^2\,\mathrm{d}x\,\mathrm{d}y$ (the rod's torsion constant)

$F \equiv \frac{1}{2}\int \tau^2\mathcal{T}\,\mathrm{d}z$ (in terms of $\mathcal{T}$)

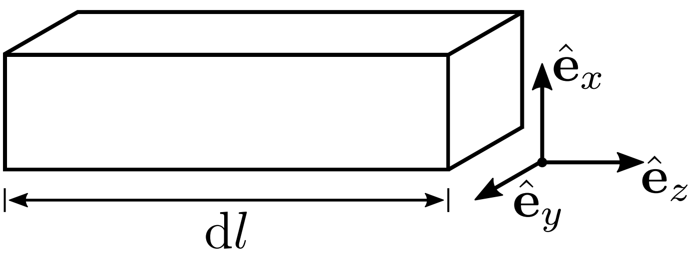

Figure 1: Coordinate system for rod flexure

Rod Flexure

Rod flexure is measured in terms of radius of curvature R and curvature $C = 1/R$.

Consider a rectangular rod segment of length $\mathrm{d}l$.

Restriction: consider only small flexure with $R \gg \mathrm{d}l$.

Let $\hat{\mathbf{n}}_\pm$ denote normals to upper and lower rod surfaces.

Use coordinate system as in figures above and below.

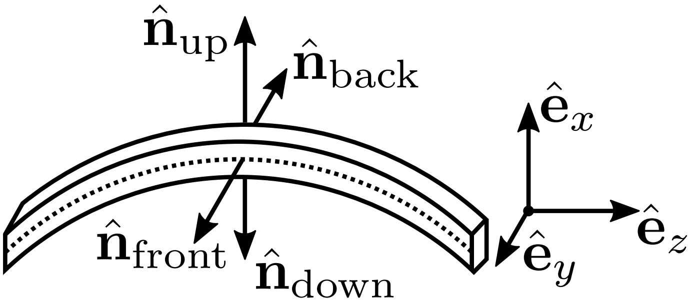

Figure 2: A flexed rod

$\hat{\mathbf{n}}_{\text{up}} \approx (0,0,+1)$ (assuming small flexure $\mathrm{d}l \ll R$)

$\hat{\mathbf{n}}_{\text{down}} \approx (0,0,-1)$ (assuming small flexure $\mathrm{d}l \ll R$)

$\hat{\mathbf{n}}_{\text{front}} \approx (0,0,+1)$ (assuming small flexure $\mathrm{d}l \ll R$)

$\hat{\mathbf{n}}_{\text{back}} \approx (0,0,-1)$ (assuming small flexure $\mathrm{d}l \ll R$)

Concept: for a rod, even small surface forces cause considerable flexure $\implies$ internal stresses from deformation are much larger than external surface forces that cause them (like for plates).

$f_i\big|_{\text{surface}} = p_{ij}n_j\big|_{\text{surface}} \approx 0$ (for negligible surface forces)

$\left[p_{xx} \approx p_{yx} \approx p_{zx} \approx 0\right]_{\text{up/down}}$ (assuming $p_{ij}n_j\big|_{\text{surface}} \approx 0$)

$\left[p_{yx} \approx p_{yy} \approx p_{yz} \approx 0\right]_{\text{front/back}}$ (assuming $p_{ij}n_j\big|_{\text{surface}} \approx 0$)

Concept: for a rod, since p must change continuously, if components $p_{ij} \approx 0$ on both rod surfaces, then $p_{ij} \approx 0$ throughout rod volume (like for plates). Resultantly....

$\left[p_{xx}, p_{yy}, p_{zz}, p_{xy}, p_{xz} \approx 0\right]_{\text{throughout rod}}$ (assuming thin rod)

$\mathsf{p} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_{zz} \end{pmatrix}$ (for a flexed rod)

Only p_{zz} is nonzero for rod flexure;

$\implies$ *Rod flexure is a uniaxial deformation!*

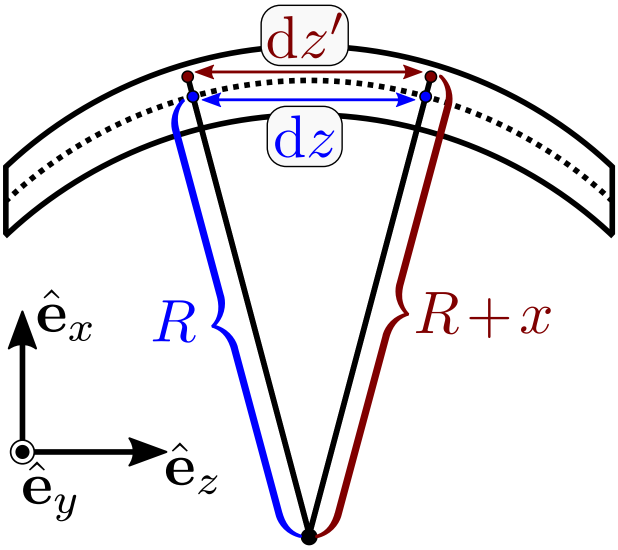

Figure 3: Geometry for finding a flexed rod's strain tensor

Strain and Stress Tensor of a Flexed Rod

Consider two pairs of points in an undeformed reference state, both separated longitudinally by $\mathrm{d}z$ and vertically by x. After deformation, the longitudinal separations are $\mathrm{d}z$ and $\mathrm{d}z'$.

$\frac{\mathrm{d}z}{R} = \frac{\mathrm{d}z'}{R+x}$ (similar triangles; see figure)

$\frac{\mathrm{d}z'}{\mathrm{d}z} = \frac{R+x}{R} = 1 + \frac{x}{R}$ (after rearranging)

$u_{zz} = \frac{\mathrm{d}z'}{\mathrm{d}z} - 1 = \frac{x}{R}$ (strain tensor component)

x is reference distance of rod element from neutral plane.
$p_{zz} = E u_{zz}$ (by definition of E via uniaxial deformation)
$u_{xx} = -\nu u_{zz}$ (by definition of ν via uniaxial deformation)
$u_{yy} = -\nu u_{zz}$ (by definition of ν via uniaxial deformation)
$u_{ij} = \frac{1}{2\mu}\left(p_{ij} - \frac{\lambda p_{kk}}{3\lambda+2\mu}\delta_{ij}\right)$ (general constitutive relation)
$\implies u_{ij} = 0$ for $i \neq j$ (because $p_{ij} = 0$ for $i \neq j$)
$\mathsf{u} = \frac{x}{R}\begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (for a flexed rod)

Displacement Vector for Rod Flexure

$u_{zz} = \frac{x}{R}$ (from above)
$u_{xx} = u_{yy} = -\nu u_{zz} = -\nu \frac{x}{R}$ (from above)
$u_x = -\nu \int \frac{x}{R}\,\mathrm{d}x = -\frac{\nu}{2}\frac{x^2}{R} + f(y,z)$ (from $u_{xx} = -\nu\frac{x}{R}$)
$u_y = -\nu \int \frac{x}{R}\,\mathrm{d}y = -\nu\frac{xy}{R} + g(x,z)$ (from $u_{yy} = -\nu\frac{x}{R}$)
$u_z = \int \frac{x}{R}\,\mathrm{d}z = \frac{xz}{R} + h(x,y)$ (from $u_{zz} = \frac{x}{R}$)
Plan: choose f, g and h to satisfy $u_{xy} = u_{xz} = u_{zy} = 0$.
$\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 0$ (from $u_{xy} = 0$)
$\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0$ (from $u_{xz} = 0$)
$\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = 0$ (from $u_{zy} = 0$)
$\implies f(y,z) = \frac{\nu}{2}\frac{y^2}{R} - \frac{z^2}{2R}$ (other possibilities exist)
$\implies g = h = 0$ (other possibilities exist)
$u_x = -\frac{1}{2R}\left[\nu(x^2 - y^2) + z^2\right]$ (in terms of $f(y,z)$)
$u_y = -\nu\frac{xy}{R}$ (using $g(x,z) = 0$)
$u_z = \frac{xz}{R}$ (using $h(x,y) = 0$)

Elastic Energy for Rod Flexure

$\mathsf{p} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E\frac{x}{R} \end{pmatrix}$ (for a flexed rod)
$\mathsf{u} = \frac{x}{R}\begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (for a flexed rod)
$p_{ij} = \frac{\partial f}{\partial u_{ij}}$ (from Work During a Deformation)
$f = \int p_{zz}\,\mathrm{d}u_{zz}$ (because only p_{zz} is nonzero)
$\quad = f_0 + \frac{E}{2R^2}x^2$ (using above u_{zz} and p_{zz})
$\quad = \frac{E}{2R^2}x^2$ (setting $f_0 = 0$)
$F = \iiint f\,\mathrm{d}V = \frac{E}{2R^2}\int \mathrm{d}z \iint x^2\,\mathrm{d}x\,\mathrm{d}y$
Let $I \equiv \iint x^2\,\mathrm{d}x\,\mathrm{d}y$. (cross-sectional moment of inertia)
I encodes the strength with which a rod resists flexure.
$F = \frac{E}{2R^2}\int I\,\mathrm{d}z$ (for a flexed rod in terms of I)

Global Rod Theory

Global rod theory analyzes rods on a macroscopic scale, parameterized as a space curve $\mathbf{r}(l)$.
l is arc length parameter along rod.
Additionally: define a local (x,y,z) coordinate system whose basis vectors are identical everywhere along the *undeformed* rod, but whose directions may change if the rod deforms.
$\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ lie in the rod's cross section.
$\hat{\mathbf{e}}_z$ points along rod's longitudinal axis.

Tangent, Normal and Binormal

$\hat{\mathbf{t}}(l) = \frac{\partial \mathbf{r}(l)}{\partial l}$ (tangent to rod)
$\hat{\mathbf{n}}(l) = \frac{\partial \hat{*}t(l)}{\partial l}\Big/\left|\frac{\partial \hat{*}t(l)}{\partial l}\right|$ (normal to rod)
$\hat{\mathbf{b}} \equiv \hat{\mathbf{t}} \times \hat{\mathbf{n}}$ (rod's binormal vector)
$C = \hat{\mathbf{n}} \cdot \frac{\partial \hat{*}t}{\partial l}$ (rod's curvature)
$R \equiv 1/C$ (radius of curvature)
$\frac{\hat{\mathbf{n}}}{R} = \frac{\partial \hat{*}t}{\partial l}$ (from definitions of $\hat{\mathbf{n}}$, R and C)
$\hat{\mathbf{t}}$ is a unit vector with magnitude $|\hat{\mathbf{t}}| = 1$, so any change in $\hat{\mathbf{t}}$ must be perpendicular to $\hat{\mathbf{t}}$ to maintain $|\hat{\mathbf{t}}| = 1 \implies \hat{\mathbf{t}} \perp \frac{\partial \hat{*}t}{\partial l}$
$\frac{\partial \hat{*}t}{\partial l} \equiv \boldsymbol{\Omega} \times \hat{\mathbf{t}}$ (motivated by $\hat{\mathbf{t}} \perp \frac{\partial \hat{*}t}{\partial l}$)
$\hat{\mathbf{t}} \times \frac{\partial \hat{*}t}{\partial l} = \hat{\mathbf{t}} \times (\boldsymbol{\Omega} \times \hat{\mathbf{t}})$ (multiplying by $\hat{\mathbf{t}}$)
$\quad = t^2\boldsymbol{\Omega} - (\boldsymbol{\Omega}\cdot\hat{\mathbf{t}})\,\hat{\mathbf{t}}$ (vector identity)
$\quad = \boldsymbol{\Omega} - (\boldsymbol{\Omega}\cdot\hat{\mathbf{t}})\,\hat{\mathbf{t}}$ (using $|\hat{\mathbf{t}}| = 1$)
$\boldsymbol{\Omega} = \hat{\mathbf{t}} \times \frac{\partial \hat{*}t}{\partial l} + (\boldsymbol{\Omega}\cdot\hat{\mathbf{t}})\,\hat{\mathbf{t}}$ (rearranging)
$\quad = \hat{\mathbf{t}} \times \frac{\hat{\mathbf{n}}}{R} + \Omega_z\,\hat{\mathbf{t}}$ (using $\frac{\hat{\mathbf{n}}}{R} = \frac{\partial \hat{*}t}{\partial l}$ and $\hat{\mathbf{t}} \parallel \hat{\mathbf{e}}_z$)
$\quad = \Omega_x\,\hat{\mathbf{e}}_x + \Omega_y\,\hat{\mathbf{e}}_y + \Omega_z\,\hat{\mathbf{e}}_z$ (writing $\boldsymbol{\Omega}$ by components)
$\Omega_z = \tau$ is torsion angle per unit rod length
$\Omega_z\,\hat{\mathbf{t}}$ is rod torsion.
$\hat{\mathbf{t}} \times \frac{\hat{\mathbf{n}}}{R} = \Omega_x\,\hat{\mathbf{e}}_x + \Omega_y\,\hat{\mathbf{e}}_y$ is rod flexure and is parallel to $\hat{\mathbf{b}}$
$\left|\hat{\mathbf{t}} \times \frac{\hat{\mathbf{n}}}{R}\right| = \frac{1}{R}$ (because $\hat{\mathbf{t}} = \hat{\mathbf{n}} = 1$)

Elastic Energy

$\left|\hat{\mathbf{t}} \times \frac{\hat{\mathbf{n}}}{R}\right|^2 = 1/R^2 = \Omega_x^2 + \Omega_y^2$ (because $\hat{\mathbf{e}}_x \perp \hat{\mathbf{e}}_y$)
Linear theory requires F is a quadratic form in $\boldsymbol{\Omega}$'s components.
We reject any odd terms in Ω_z, i.e. $\Omega_z\Omega_i$ terms for $i \neq z$, since these terms non-physically change sign under $\hat{\mathbf{e}}_z \to -\hat{\mathbf{e}}_z$.
$\implies F = \frac{1}{2}\int\left[E(I_{yy}\Omega_y^2 + 2I_{yx}\Omega_y\Omega_x + I_{xx}\Omega_x^2) + \mathcal{T}\Omega_z^2\right]\mathrm{d}l$
E is rod's Young's modulus.
$I_{ij} = \iint x_i x_j\,\mathrm{d}x\,\mathrm{d}y$ are cross-sectional moments of inertia
$\mathcal{T} = 4\mu\iint(\nabla\chi)^2\,\mathrm{d}x\,\mathrm{d}y$ is rod's torsion constant
μ is rod's shear modulus.
χ is rod's associated torsion function.
In the rod's system of principal axes, the mixed terms vanish, leaving...
$F = \frac{1}{2}\int\left[E(I_1\Omega_y^2 + I_2\Omega_x^2) + \mathcal{T}\Omega_z^2\right]\mathrm{d}l$ (in rod's system of PA)

Limit of Small Flexure and No Torsion

$\mathbf{r}(l) \approx \big(X(l), Y(l), l\big)$ (ansatz; $r_z \approx l$ for small flexure)
$\hat{\mathbf{t}} = \frac{\partial \mathbf{r}}{\partial l} \approx \big(X'(l), Y'(l), 1\big)$
$\boldsymbol{\Omega} = \hat{\mathbf{t}} \times \frac{\hat{\mathbf{n}}}{R} + \tau\,\hat{\mathbf{t}} = \Omega_x\,\hat{\mathbf{e}}_x + \Omega_y\,\hat{\mathbf{e}}_y + \tau\,\hat{\mathbf{t}}$ (in general)
$\quad = \hat{\mathbf{t}} \times \frac{\hat{\mathbf{n}}}{R} = \hat{\mathbf{t}} \times \frac{\partial\hat{\mathbf{t}}}{\partial l}$ (no torsion; using $\frac{\hat{\mathbf{n}}}{R} = \frac{\partial \hat{*}t}{\partial l}$)
$\quad \approx \big(-Y''(l), X''(l), 0\big)$ (neglecting $\mathcal{O}(XY)$ terms)
$F \approx \frac{E}{2}\int(I_{yy}X''^2 + 2I_{yx}X''Y'' + I_{xx}Y''^2)\,\mathrm{d}l$ (in the limit of small flexure and no torsion)
$\quad \approx \frac{E}{2}\int\left(I_1 Y''^2 + I_2 X''^2\right)\mathrm{d}l$ (in rod's system of PA)

Equilibrium of Forces

Consider infinitesimal rod segment of length $\mathrm{d}l$.
$\mathbf{F} + \mathrm{d}\mathbf{F}$ is contact force on cross-sectional surface with larger l.
$-\mathbf{F}$ is contact force on cross-sectional surface with smaller l.
$\mathrm{d}\mathbf{F}$ is net internal stress-related force on rod segment.
$\mathbf{K}$ is linear force density of external forces (e.g. weight).
$\mathrm{d}\mathbf{F} + \mathbf{K}\,\mathrm{d}l = \mathbf{0}$ (equilibrium condition for rod segment)
$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}l} = -\mathbf{K}$ (equilibrium condition, rearranged)
$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}l} = \mathbf{0} \implies \mathbf{F} = \text{constant}$ (if $\mathbf{K} = \mathbf{0}$)

Equilibrium of Torques

Consider infinitesimal rod segment of length $\mathrm{d}l$.
Define torques on segment's cross sections with respect to cross sections' centroids.
$\mathbf{M} + \mathrm{d}\mathbf{M}$ is contact torque on cross section with larger l.
$-\mathbf{M}$ is contact torque on cross section with smaller l.
$\mathrm{d}\mathbf{M}$ is net internal stress-related torque on rod segment.
$-\mathrm{d}\mathbf{l} \times (-\mathbf{F})$ is torque from internal stress forces.
Simplification: neglect $\mathcal{O}(\mathrm{d}l^2)$ torque from external forces.
$\mathrm{d}\mathbf{M} + \mathrm{d}\mathbf{l} \times \mathbf{F} = \mathbf{0}$ (equilibrium condition for rod segment).
Let $\hat{\mathbf{t}} \equiv \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}l}$ (unit tangent vector to rod)
$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}l} = \mathbf{F} \times \hat{\mathbf{t}}$ (equilibrium condition in terms of $\hat{\mathbf{t}}$)

Miscellaneous Notes

$\hat{\mathbf{t}}' = \frac{\mathrm{d}\,\hat{*}t}{\mathrm{d}l}$ is rod's curvature
$I_{ij} = \iint(r^2\delta_{ij} - r_i r_j)\,\mathrm{d}S$; $\mathbf{r} = (x,y)$ (area moment of inertia)

Boundary Conditions

$\mathbf{r} = \text{constant}$ at a fixed end.
$\hat{\mathbf{t}} = \text{constant}$ at a fixed end.

$\mathbf{r} =$ constant at a supported end.
$\hat{\mathbf{t}} \neq$ constant at a supported end.
$\mathbf{M} = \mathbf{0}$ along rod's longitudinal axis in equilibrium.
$\mathbf{F} = \mathbf{0}$ at rod's free end.
$\mathbf{M} = \mathbf{0}$ at rod's free end.

Kirchhoff Theory (Circular Rod)

Consider rod with a circular cross section for which $I_1 = I_2 \equiv I$.
Perform analysis in rod's system of principal axes.
$F = \frac{1}{2}\int \left[EI(\Omega_y^2 + \Omega_x^2) + C\Omega_z^2\right] \mathrm{d}l$ (rod's free energy)
$\mathbf{M} = (EI\Omega_x, EI\Omega_y, \mathcal{T}\Omega_z)$ (ansatz for stress torque)
M_x and M_y are associated with flexure
M_z is associated with torsion
$\boldsymbol{\Omega} = \hat{\mathbf{t}} \times \frac{\hat{\mathbf{n}}}{R} + \Omega_z \hat{\mathbf{t}}$ (from Tangent, Normal and Binormal)
$\quad = \hat{\mathbf{t}} \times \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}l} + \Omega_z \hat{\mathbf{t}}$ (using $\frac{\hat{\mathbf{n}}}{R} = \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}l}$)
$\mathbf{M} = EI\,\hat{\mathbf{t}} \times \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}l} + \mathcal{T}\Omega_z \hat{\mathbf{t}}$ (torque-deformation relationship)
$\mathbf{M} = EI\,\hat{\mathbf{t}} \times \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}l}$ (torsion-free deformation; $\Omega_z = \tau = 0$)
$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}l} = EI\left(\frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}l} \times \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}l} + \hat{\mathbf{t}} \times \frac{\mathrm{d}^2\hat{\mathbf{t}}}{\mathrm{d}l^2}\right)$
$\quad = EI\,\hat{\mathbf{t}} \times \frac{\mathrm{d}^2\hat{\mathbf{t}}}{\mathrm{d}l^2}$ (because $\frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}l} \times \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}l} = \mathbf{0}$)
$\mathbf{F} \times \hat{\mathbf{t}} = EI\,\hat{\mathbf{t}} \times \frac{\mathrm{d}^2\hat{\mathbf{t}}}{\mathrm{d}l^2}$ (using $\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}l} = \mathbf{F} \times \hat{\mathbf{t}}$)

Linear Equations for Small Flexure, No Torsion

See also Limit of Small Flexure and No Torsion.
$\frac{\mathrm{d}}{\mathrm{d}z} \approx \frac{\mathrm{d}}{\mathrm{d}l}$ (small flexure)
$\hat{\mathbf{t}} \approx \left(\frac{\mathrm{d}x}{\mathrm{d}z}, \frac{\mathrm{d}y}{\mathrm{d}z}, 1\right) \equiv (\dot{x}, \dot{y}, 1)$ (small flexure)
$\frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}z} \approx \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}l} \approx \left(\frac{\mathrm{d}^2x}{\mathrm{d}z^2}, \frac{\mathrm{d}^2y}{\mathrm{d}z^2}, 0\right) \equiv (\ddot{x}, \ddot{y}, 0)$ (small flexure)
Simplification: assume a torsion-free deformation.
$\mathbf{M} = EI\,\hat{\mathbf{t}} \times \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}l}$ (general torsion-free deformation)
$\quad \approx EI\,\hat{\mathbf{t}} \times \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}z}$ (small flexure, $\frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}z} \approx \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}l}$)
$\quad = EI(-\ddot{y}, \ddot{x}, 0)$ (neglecting $\mathcal{O}(x_j^2)$ terms)
$\mathbf{F} \times \hat{\mathbf{t}} \approx (F_y - \dot{y}F_z, \dot{x}F_z - F_x, \dot{y}F_x - \dot{x}F_y)$ (small flexure)
$\quad \approx (F_y - \dot{y}F_z, \dot{x}F_z - F_x, 0)$ (neglecting z component)
$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}l} = \mathbf{F} \times \hat{\mathbf{t}}$ (general torque equilibrium condition)
$EI(-\dddot{y}, \dddot{x}, 0) = (F_y - \dot{y}F_z, \dot{x}F_z - F_x, 0)$ (for small flexure)
$\implies EIy^{(3)} = \dot{y}F_z - F_y$ (small flexure)
$\implies EIx^{(3)} = \dot{x}F_z - F_x$ (small flexure)
$EIy^{(4)} = \ddot{y}F_z + \dot{y}\dot{F}_z - \dot{F}_y$ (after differentiating)
$EIx^{(4)} = \ddot{x}F_z + \dot{x}\dot{F}_z - \dot{F}_x$ (after differentiating)
$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}l} = -\mathbf{K}$ (general force equilibrium condition)
$\implies \frac{\mathrm{d}F_i}{\mathrm{d}z} \approx \frac{\mathrm{d}F_i}{\mathrm{d}l} = -K_i$ (for small flexure)
$EIy^{(4)} = \ddot{y}F_z + \dot{y}\dot{F}_z + K_y$ (in terms of $\mathbf{K}$)
$EIx^{(4)} = \ddot{x}F_z + \dot{x}\dot{F}_z + K_x$ (in terms of $\mathbf{K}$)

Dynamics

$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}l} + \mathbf{K} = \mu\frac{\partial^2\mathbf{r}}{\partial t^2}$
$EIy^{(4)} + \mu\frac{\partial^2 y}{\partial t^2} = \ddot{y}F_z + \dot{y}\dot{F}_z + K_y$
$EIx^{(4)} + \mu\frac{\partial^2 x}{\partial t^2} = \ddot{x}F_z + \dot{x}\dot{F}_z + K_x$
Oscillation ansatz: $u(z,t) = v(z)e^{-i\omega t}$

Elastic Waves

Waves are deformations traveling through a body.
Concept: body elements moving relative to each other during deformation cause temperature change in body, leading to heat flow through body.
Restriction: we consider only adiabatic situations in which heat flows slowly enough that constituent elements of a continuous body are approximately thermally isolated from their neighbors.
Concept: assuming adiabatic processes, we can analyze waves with thus-far-derived *isothermal* relationships, as long as we replace isothermal quantities with their *adiabatic* analogs.
Notation: all elastic constants in this section are assumed to be implicitly adiabatic, e.g. E means E_S (and not E_T).

Longitudinal and Transverse Polarizations

Consider plane waves in an unbounded isotropic material.
Align coordinate system so waves travel along x axis.
$\mathbf{u} = \mathbf{u}(x,t)$ (displacement vector in material)
$\rho\ddot{\mathbf{u}} = \frac{E}{2(1+\nu)}\left[\nabla^2\mathbf{u} + \frac{1}{1-2\nu}\nabla(\nabla\cdot\mathbf{u})\right]$ (Navier equation)
$\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{\partial^2 u_x}{\partial x^2}$ (for x component)
Let $c_\mathrm{l}^2 \equiv \frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}$ (longitudinal wave speed)
(Longitudinal because direction of displacement $u_x = \mathbf{u}\cdot\hat{\mathbf{e}}_x$ matches wave propagation direction $\hat{\mathbf{e}}_x$.)
$\frac{\partial^2 u_x}{\partial t^2} = c_\mathrm{l}^2\frac{\partial^2 u_x}{\partial x^2}$ (x component eq. in terms of c_l)
$\rho\frac{\partial^2 u_y}{\partial t^2} = \frac{E}{2(1+\nu)}\frac{\partial^2 u_y}{\partial x^2}$ (for y component)
$\rho\frac{\partial^2 u_z}{\partial t^2} = \frac{E}{2(1+\nu)}\frac{\partial^2 u_z}{\partial x^2}$ (for z component)
Let $c_\mathrm{t}^2 \equiv \frac{E}{2\rho(1+\nu)}$. (transverse wave speed)
(Transverse because directions of displacement $u_y = \mathbf{u}\cdot\hat{\mathbf{e}}_y$ and $u_z = \mathbf{u}\cdot\hat{\mathbf{e}}_z$ are transverse to wave propagation direction $\hat{\mathbf{e}}_x$.)
$\frac{\partial^2 u_y}{\partial t^2} = c_\mathrm{t}^2\frac{\partial^2 u_y}{\partial x^2}$ (y component eq. in terms of c_t)
$\frac{\partial^2 u_z}{\partial t^2} = c_\mathrm{t}^2\frac{\partial^2 u_z}{\partial x^2}$ (z component eq. in terms of c_t)
Conclusion: elastic waves in an isotropic material have one longitudinal polarization and two transverse polarizations.
$\nabla \times \mathbf{u}_\mathrm{l} = \mathbf{0}$ (in general for longitudinal polarizations)
$\nabla \cdot \mathbf{u}_\mathrm{t} = 0$ (in general for transverse polarizations)

Interpreting Wave Speeds

$c_\mathrm{l}^2 = \frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}$ (longitudinal wave speed)
$c_\mathrm{t}^2 = \frac{E}{2\rho(1+\nu)}$ (transverse wave speed)
$c_\mathrm{l}/c_\mathrm{t} = \sqrt{\frac{2(1-\nu)}{(1-2\nu)}}$ (ratio of wave speeds)
$c_\mathrm{l} > c_\mathrm{t}$ for physical materials with $\nu \in (-1, 1/2)$
$c_\mathrm{t}^2 = \frac{\mu}{\rho}$ (using $\mu = \frac{E}{2(1+\nu)}$)
Conclusion: $c_\mathrm{t} = 0$ in materials with $\mu = 0$, i.e. materials (e.g. fluids) that cannot transfer shear stress.
$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ (from Relationship Btwn. Elastic Constants)
$\mu = \frac{E}{2(1+\nu)}$ (from Relationship Btwn. Elastic Constants)
$c_\mathrm{l}^2 = \frac{\lambda + 2\mu}{\rho}$ (using above relationships)
$\quad = \frac{K + 4\mu/3}{\rho}$ (using $K = \lambda + \frac{2\mu}{3}$)
$c_\mathrm{l}^2 = K/\rho$ in materials (e.g. fluids) with $\mu = 0$; note that $c_\mathrm{l} = \sqrt{K/\rho}$ agrees with the speed of sound in fluids.

Reflection and Refraction

Consider waves incident on the boundary between two materials.
Displacement vector and force components must change continuously across the boundary.
Wave frequency ω is conserved across the boundary.
Wave vector $\mathbf{k}$'s tangential component to boundary is conserved across boundary.

Geometry and Coordinate System

Consider boundary in yz plane between materials 1 and 2.
Let plane of incidence (plane containing incident, reflected and transmitted wave vectors) align with xy plane.
$\implies$ inc., ref., and trans. wave vectors all have $k_z = 0$.
$\mathbf{k}_\parallel = k_y$ (component of $\mathbf{k}$ $\parallel$ to boundary for xy PoI)
Let c_1 denote wave speed in material 1.
Let c_2 denote wave speed in material 2.
c_1 and c_2 vary depending on wave polarization!.
Subscript $_\mathrm{i}$ refers to incident quantities.
Subscript $_\mathrm{r}$ refers to reflected quantities.
Subscript $_\mathrm{t}$ refers to transmitted quantities.
$\mathbf{k}_\mathrm{i} = \frac{\omega}{c_1^{(\mathrm{i})}}(\cos\theta_\mathrm{i}, \sin\theta_\mathrm{i}, 0)$ (incident wave vector)
$\mathbf{k}_\mathrm{r} = \frac{\omega}{c_1^{(\mathrm{r})}}(-\cos\theta_\mathrm{r}, \sin\theta_\mathrm{i}, 0)$ (reflected wave vector)

$\mathbf{k}_\mathrm{i} = \frac{\omega}{c_2^{(\mathrm{t})}}(\cos\theta_\mathrm{t}, \sin\theta_\mathrm{t}, 0)$ (transmitted wave vector)

Conservation Laws

$\omega_\mathrm{i} = \omega_\mathrm{r} = \omega_\mathrm{t}$ (frequency is preserved)

$k_\mathrm{i}^{\|} = k_\mathrm{r}^{\|} = k_\mathrm{t}^{\|}$ (longitudinal component $\mathbf{k}_\|$ preserved)

$\implies k_{\mathrm{i}_y} = k_{\mathrm{r}_y} = k_{\mathrm{t}_y}$ (for xy PoI for which $\mathbf{k}_\| = k_y$)

Law of Reflection

$k_{\mathrm{i}_y} = k_{\mathrm{r}_y}$ (longitudinal component $\mathbf{k}_\|$ preserved)

$\frac{\omega}{c_1^{(\mathrm{i})}} \sin\theta_\mathrm{i} = \frac{\omega}{c_1^{(\mathrm{i})}} \sin\theta_\mathrm{r}$ (from $k_{\mathrm{i}_y} = k_{\mathrm{r}_y}$)

$\implies c_1^{(\mathrm{r})} \sin\theta_\mathrm{i} = c_1^{(\mathrm{i})} \sin\theta_\mathrm{r}$ (law of reflection)

$c_1^{(\mathrm{i})} = c_1^{(\mathrm{r})}$ if incident and reflected waves have equal polarizations.

$\implies \theta_\mathrm{i} = \theta_\mathrm{r}$ (if $c_1^{(\mathrm{i})} = c_1^{(\mathrm{r})}$)

Law of Refraction

$k_{\mathrm{i}_y} = k_{\mathrm{t}_y}$ (longitudinal component $\mathbf{k}_\|$ preserved)

$\frac{\omega}{c_1^{(\mathrm{i})}} \sin\theta_\mathrm{i} = \frac{\omega}{c_2^{(\mathrm{t})}} \sin\theta_\mathrm{t}$ (from $k_{\mathrm{i}_y} = k_{\mathrm{t}_y}$)

$\implies c_2^{(\mathrm{t})} \sin\theta_\mathrm{i} = c_1^{(\mathrm{i})} \sin\theta_\mathrm{t}$ (law of refraction)

Surface Waves

Consider elastic waves propagating in the $\hat{\mathbf{e}}_x$ direction in a material filling the region $z < 0$.

$\mathbf{u} = \mathbf{u}(x, z, t)$ (displacement vector in material)

$u_i(x, z, t) = f(z) e^{i(kx - \omega t)}$ (ansatz for components u_i)

$f(z)$ is a function encoding surface effects.

$\ddot{\mathbf{u}} = c^2 \nabla^2 \mathbf{u}$ (general wave equation)

$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} = \left(k^2 - \frac{\omega^2}{c^2}\right) f(z)$ (substituting ansatz into wave eq.)

Let $\kappa^2 \equiv k^2 - \omega^2/c^2$

$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} = \kappa^2 f$ (in terms of κ)

$\implies f(z) \propto e^{\kappa z}$

Note: c and thus κ depend on wave polarization.

Boundary Conditions

Decompose $\mathbf{u}$ into longitudinal and transverse components with respect to boundary surface.

$\mathbf{u} = \mathbf{u}_\mathrm{l} + \mathbf{u}_\mathrm{t}$ (decomposition of $\mathbf{u}$)

$\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{t}$ are related by boundary conditions at the surface.

Let $\hat{\mathbf{n}} = (0, 0, 1)$ denote normal to material surface $z = 0$.

$\left[f_i = p_{ij} n_j = 0\right]_{z=0}$ (no stress on surface)

$p_{xz} = 0$ (from f_x component)

$p_{yz} = 0$ (from f_y component)

$p_{zz} = 0$ (from f_z component)

$p_{ij} = \lambda u_{kk} \delta_{ij} + 2\mu u_{ij}$ (Hooke's law, for review)

$\implies u_{xz} = 0$ and $u_{yz} = 0$ (from $p_{xz} = 0$ and $p_{yz} = 0$)

$\mathbf{u} = \mathbf{u}(x, z, t) \implies \mathbf{u} \neq \mathbf{u}(y) \implies \frac{\partial u_i}{\partial y} = 0$

$\implies \frac{\partial u_y}{\partial z} = 0$ (from $u_{yz} = 0$ and $\frac{\partial u_z}{\partial y} = 0$)

$\implies u_y = 0$ or $u_y = \text{constant}$ (from $\frac{\partial u_z}{\partial y} = 0$)

Idea: for a wave (which oscillates), $u_i \neq \text{constant}$.

$\implies u_y = 0$ (from $u_y \neq \text{constant}$)

Conclusion: displacement vector $\mathbf{u}$ must lie in xz plane.

Transverse Polarization

$u_{\mathrm{t}_\alpha} \propto e^{\kappa_\mathrm{t} z} e^{i(kx - \omega t)}$ (ansatz for transverse polarization)

$\nabla \cdot \mathbf{u}_\mathrm{t} = 0$ (in general for transverse polarization)

$\frac{\partial u_{\mathrm{t}_x}}{\partial x} + \frac{\partial u_{\mathrm{t}_z}}{\partial z} = 0$ (because $u_y = 0$)

$iku_{\mathrm{t}_x} + \kappa_\mathrm{t} u_{\mathrm{t}_z} = 0$ (using above ansatz)

$u_{t_x} = \kappa_t a e^{i(kx - \omega t)} e^{\kappa_t z}$ (to satisfy $\nabla \cdot \mathbf{u}_\mathrm{t} = 0$)

$u_{t_z} = -ika e^{i(kx - \omega t)} e^{\kappa_t z}$ (to satisfy $\nabla \cdot \mathbf{u}_\mathrm{t} = 0$)

a is an amplitude term for transverse polarization.

Longitudinal Polarization

$u_{\mathrm{l}_\alpha} \propto e^{\kappa_\mathrm{l} z} e^{i(kx - \omega t)}$ (ansatz for longitudinal polarization)

$\nabla \times \mathbf{u}_\mathrm{l} = \mathbf{0}$ (in general for longitudinal polarization)

$\left(\frac{\partial u_{\mathrm{l}_x}}{\partial z} - \frac{\partial u_{\mathrm{l}_z}}{\partial x}\right) \hat{\mathbf{e}}_y = \mathbf{0}$ (because $u_y = 0$ and $\mathbf{u} \neq \mathbf{u}(y)$)

$\frac{\partial u_{\mathrm{l}_x}}{\partial z} = \frac{\partial u_{\mathrm{l}_z}}{\partial x}$ (rearranging above condition)

$\kappa_\mathrm{l} u_{\mathrm{l}_x} = iku_{\mathrm{l}_z}$ (using above ansatz)

$u_{l_x} = kb e^{i(kx - \omega t)} e^{\kappa_\mathrm{l} z}$ (to satisfy $\nabla \times \mathbf{u}_\mathrm{l} = v_0$)

$u_{l_z} = -i\kappa_l b e^{i(kx - \omega t)} e^{\kappa_\mathrm{l} z}$ (to satisfy $\nabla \times \mathbf{u}_\mathrm{l} = v_0$)

b is an amplitude term for longitudinal polarization.

Fluid Dynamics

We will model fluids as *continuous* matter. Fluid motion then refers to motion of continuous volume elements of fluid and not motion of discrete fluid molecules.
We treat fluid elements are infinitesimally small relative to entire fluid but still large relative to the size of fluid molecules.
The state of a moving fluid is by the velocity, pressure and density fields $\mathbf{v}$, p and ρ.
Field values refer to a given position in space and time and not to a particular fluid element moving through space.

Conservation of Mass

A *control volume* is a small fixed region in space in which we observe fluid flow. Analogy: cage with perfectly porous walls.
$\hat{\mathbf{n}}$ denotes the outward normal to the control volume's surface.
Q_m denotes mass flow rate (SI units $\mathrm{kg\,s^{-1}}$).
Q_V denotes volume flow rate (SI units $\mathrm{m^3\,s^{-1}}$).

Consider fluid with velocity $\mathbf{v}$ and density ρ in control volume V_0.
$m = \iiint_{V_0} \rho \,\mathrm{d}V$ (mass of fluid in control volume)
$\mathbf{j}_\mathrm{m} = \rho\mathbf{v}$ (mass current density/mass flux)
$\mathrm{d}Q_\mathrm{m} = \mathbf{j}_\mathrm{m} \cdot \mathrm{d}\mathbf{S}$ (MFR through surface element $\mathrm{d}\mathbf{S}$)
$Q_\mathrm{m} = \oiint_{\partial V_0} \mathbf{j}_\mathrm{m} \cdot \mathrm{d}\mathbf{S}$ (MFR through surface of V_0)

Continuity Equation for Mass

Consider fluid with velocity $\mathbf{v}$ and density ρ in control volume V_0.
$\frac{\partial m}{\partial t} + Q_\mathrm{m} = 0$ (conservation of mass)
$\iiint \frac{\partial \rho}{\partial t}\,\mathrm{d}V + \oiint \mathbf{j}_\mathrm{m} \cdot \mathrm{d}\mathbf{S} = 0$ (in integral form)
$\iiint \frac{\partial \rho}{\partial t}\,\mathrm{d}V + \iiint \nabla \cdot \mathbf{j}_\mathrm{m}\,\mathrm{d}V = 0$ (using divergence theorem)
$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}_\mathrm{m} = 0$ (**mass continuity equation**)
$\frac{\partial \rho}{\partial t} + \rho\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla\rho = 0$ (cont. eq. in terms of ρ and $\mathbf{v}$)

Continuity Equation for Fluid Momentum

$\mathbf{f}$ is external force density.
$\boldsymbol{\pi}$ is fluid momentum density.
$\Pi_{ij} = \rho v_i v_j + p\delta_{ij}$ is momentum current density.
$\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j}\Pi_{ij} = f_i$ (**momentum continuity equation**)

Ideal Fluid Flow

Ideal flow: no energy dissipation due to internal viscous (friction) forces; no exchange of heat between the constituent fluid elements; no shear stress.
Inviscid flow: flow in which fluid viscosity is zero.
Adiabatic flow: no transfer of heat between neighboring fluid elements.
Ideal flow is inviscid and adiabatic.

Dynamics of a Fluid Element

Consider ideal fluid flow with velocity $\mathbf{v}$, pressure p and density ρ in a control volume V_0.
$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \iiint_{V_0} \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,\mathrm{d}V = \mathbf{F}$ (Newton's law for fluid in V_0)
$\mathbf{F} = -\oiint_{\partial V_0} p\,\mathrm{d}\mathbf{S} + \iiint_{V_0} \rho\mathbf{g}\,\mathrm{d}V$ (force on fluid in V_0)
$\quad = \iiint_{V_0} (\rho\mathbf{g} - \nabla p)\,\mathrm{d}V$ (using scalar divergence theorem)
$\iiint \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,\mathrm{d}V = \iiint (\rho\mathbf{g} - \nabla p)\,\mathrm{d}V$ (Newton's law for fluid in V_0)
$\implies \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho\mathbf{g} - \nabla p$ (equating integrands)

Material Derivative of Velocity

Acceleration $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$ in above dynamics has two contributions:

(i) intrinsic change in $\mathbf{v}$ at a given point in space independent of presence of fluid flow, and

(ii) convective change in $\mathbf{v}$ as fluid element moves through space because of fluid flow.

$\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ depends explicitly on both position and time.
The *total* time derivative of $\mathbf{v}$ is thus...
$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(\mathbf{r}, t) = \frac{\mathrm{d}t}{\mathrm{d}t}\frac{\partial \mathbf{v}}{\partial t} + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{r}}$ (multivariable chain rule)
$\quad = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{v}$ (using $\frac{\partial \mathbf{v}}{\partial \mathbf{r}} = \nabla\mathbf{v}$)
The resulting operation $\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)$ is given a special name...
$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} \equiv \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}$ (**material derivative**)

Euler Equation

$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho\mathbf{g} - \nabla p$ (from Dynamics of a Fluid Element)
$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{g} - \frac{\nabla p}{\rho}$ (after rearranging)
$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \to \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} \equiv \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}$ (because $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$)
Replace $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$ with $\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}$ in above dynamics equation and get...
$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{g} - \frac{\nabla p}{\rho}$ (**Euler equation**)
The Euler eq. applies only to ideal (inviscid and adiabatic) flow!

Thermodynamics Review I

U is internal energy.
H is enthalpy.
S is entropy.
T is temperature.
Notation: lowercase quantities denote mass-specific quantities, i.e. quantities per unit fluid mass. Example: S is entropy; s is entropy per unit fluid mass.
$\mathrm{d}U = T\,\mathrm{d}S - p\,\mathrm{d}V$ (fundamental thermodynamic relation)
$H = U + pV$ (definition of enthalpy)
$\implies \mathrm{d}H = T\,\mathrm{d}S + V\,\mathrm{d}p$
$\mathrm{d}h = T\,\mathrm{d}s + \mathrm{d}p/\rho$ (specific enthalpy)
$G = H - TS$ (definition of Gibbs free energy)
$\implies \mathrm{d}G = -S\,\mathrm{d}T + V\,\mathrm{d}p$
$\mathrm{d}\tilde{g} = -s\,\mathrm{d}T + \mathrm{d}p/\rho$ (specific Gibbs free energy)

Euler Equation for Isentropic Flow

Isentropic flow: entropy of flow is constant (isentropic flow is both adiabatic and reversible).
$\mathrm{d}S = 0 \implies \mathrm{d}s = 0$ (for isentropic processes)
$\mathrm{d}h = T\,\mathrm{d}s + \mathrm{d}p/\rho$ (specific enthalpy in general)
$\mathrm{d}h = \mathrm{d}p/\rho$ (for isentropic processes)
$\implies \nabla h = \frac{\nabla p}{\rho}$ (for isentropic processes)
$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{g} - \frac{\nabla p}{\rho}$ (Euler equation in general)
$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{g} - \nabla h$ (Euler equation for isentropic flow)

Isentropic Euler Equation in Gradient-Free Form

Assume inviscid and isentropic flow.
$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{g} - \nabla h$ (isentropic Euler equation)
$(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla v^2 - \mathbf{v} \times (\nabla \times \mathbf{v})$ (vector calculus identity)
Use above identity to rewrite isentropic Euler equation as...
$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{g} - \nabla\left(h - \frac{v^2}{2}\right)$
$\nabla \times (\nabla\phi) = \mathbf{0}$ (identity for any scalar field ϕ)
Take curl of above equation and apply $\nabla \times (\nabla\phi) = \mathbf{0}$ to get...
$\frac{\partial \nabla \times \mathbf{v}}{\partial t} = \nabla \times \left[\mathbf{v} \times (\nabla \times \mathbf{v})\right]$ (gradient-free Euler eq.)

Hydrostatics

TODO: stress difference between mechanical and TD equilibrium
Hydrostatics is the study of fluid in equilibrium.
$\mathbf{v} = \mathbf{0}$ for fluids in mechanical equilibrium.
$\mathrm{d}T = 0$ for fluids in thermodynamic equilibrium.
$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{g} - \frac{\nabla p}{\rho}$ (Euler equation in general)
$\frac{\nabla p}{\rho} = \mathbf{g}$ (Euler equation for hydrostatics)
$\mathrm{d}\tilde{g} = -s\,\mathrm{d}T + \mathrm{d}p/\rho$ (specific Gibbs free energy in general)
$\mathrm{d}\tilde{g} = \mathrm{d}p/\rho$ (for hydrostatics where $\mathrm{d}T = 0$)
$\implies \nabla\tilde{g} = \frac{\nabla p}{\rho}$ (in hydrostatic equilibrium)
$\mathbf{g} = \nabla\tilde{g}$ (Euler equation in hydrostatics)
Align coordinate system so $\hat{\mathbf{e}}_z \parallel -\mathbf{g}$.

$\mathbf{g} = -g\nabla z = -\nabla(gz)$ (if $\hat{\mathbf{e}}_z \parallel -\mathbf{g}$ and g is constant)
$\nabla(\tilde{g} + gz) = \mathbf{0}$ (hydrostatic Euler eq. if $\hat{\mathbf{e}}_z \parallel \mathbf{g}$)
$\implies \tilde{g} + gz = \text{constant}$ (if $\hat{\mathbf{e}}_z \parallel \mathbf{g}$)
$\nabla p = \rho\mathbf{g}$ (general hydrostatic situation)
$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g$ (if $\hat{\mathbf{e}}_z \parallel \mathbf{g}$)

Convection

(Thermal) convection is spontaneous fluid flow, caused by a large temperature gradient, that mixes fluid back into a state of thermal equilibrium.
Notation: for this section on convection only, let $v \equiv V/m$ denote specific volume (fluid volume per unit mass) and not magnitude of fluid velocity.
$v = V/m = 1/\rho$ (various formulations of specific volume)

Equilibrium Condition

Goal: find condition at which convection is absent (and thus the condition at which mechanical equilibrium is steady).
Align coordinate system so that $\hat{\mathbf{e}}_z \parallel -\mathbf{g}$.
$v = v(p, s)$ (v depends on pressure and specific entropy)
p and s both depend on vertical coordinate z.
Consider element of *isentropic* fluid with specific entropy s_0 originally at reference position z_0 where the pressure is p_0.
Consider a second fluid element at higher position $z = z_0 + \Delta z$ (where $\Delta z > 0$) with specific entropy s and pressure p.

Imagine lower fluid element (LFE) moving up and displacing upper fluid element (UFE).
$v(p_0, s_0) \to v(p, s_0)$ (change in LFE's specific entropy)
(s_0 is constant assuming isentropic fluid)
LFE returns to original position z_0 (maintaining equilibrium) if LFE has smaller specific volume at $z = z_0 + \Delta z$ than displaced UFE originally at z, i.e. if...
$v_{\text{displacing}} < v_{\text{displaced}}$ (equilibrium condition)
$v(p, s_0) < v(p, s)$ (equivalent formulation)

Thermodynamics Review II

c_p is fluid's specific heat capacity at constant pressure.
$\mathrm{d}S = \frac{mc_\mathrm{p}}{T}\,\mathrm{d}T + \left(\frac{\partial S}{\partial p}\right)_T \mathrm{d}p$
$\mathrm{d}S = \frac{mc_\mathrm{p}}{T}\,\mathrm{d}T$ (in isobaric situations with $\mathrm{d}p = 0$)
$\implies \left(\frac{\partial S}{\partial v}\right)_p = \frac{mc_\mathrm{p}}{T}\left(\frac{\partial T}{\partial v}\right)_p$
$\left(\frac{\partial s}{\partial v}\right)_p = \frac{c_\mathrm{p}}{T}\left(\frac{\partial T}{\partial v}\right)_p$ (in terms of specific quantities)
$\left(\frac{\partial v}{\partial s}\right)_p = \frac{T}{c_\mathrm{p}}\left(\frac{\partial v}{\partial T}\right)_p$ (after rearranging)

Analyzing Equilibrium Condition I

$v(p, s) - v(p, s_0) > 0$ (equilibrium condition)
Expand $v(p, s)$ about $v(p, s_0)$ to first order in $\frac{\mathrm{d}s}{\mathrm{d}z}\Delta z$ to get...
$v(p, s) = v(p, s_0) + \left(\frac{\partial v}{\partial s}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z}\Delta z + \mathcal{O}\left(\left(\frac{\mathrm{d}s}{\mathrm{d}z}\right)^2 (\Delta z)^2\right)$
$\left(\frac{\partial v}{\partial s}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z}\Delta z > 0$ (equilibrium condition to $\mathcal{O}(\Delta z)$)
$\left(\frac{\partial v}{\partial s}\right)_p = \frac{T}{c_\mathrm{p}}\left(\frac{\partial v}{\partial T}\right)_p$ (from Thermodynamics Review II)
$\frac{T\Delta z}{c_\mathrm{p}}\left(\frac{\partial v}{\partial T}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z} > 0$ (eq. condition using above identity)
$\implies \left(\frac{\partial v}{\partial T}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z} > 0$ (because $\frac{T\Delta z}{c_\mathrm{p}}$ is always positive)

Thermodynamics Review III

α is fluid's coefficient of thermal expansion.
K is fluid's bulk modulus (resistance to bulk deformations).
$\frac{\mathrm{d}V}{V} = \alpha\,\mathrm{d}T - \frac{\mathrm{d}p}{K}$ (general thermodynamics identity)
$\left(\frac{\partial V}{\partial T}\right)_p = \alpha V$ (in isobaric situations with $\mathrm{d}p = 0$)
$\left(\frac{\partial v}{\partial T}\right)_p = \alpha v$ (in terms of specific volume)

Analyzing Equilibrium Condition II

$\left(\frac{\partial v}{\partial T}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z} > 0$ (equilibrium condition)
$\left(\frac{\partial v}{\partial T}\right)_p = \alpha v$ (from Thermodynamics Review III)
$\alpha v \frac{\mathrm{d}s}{\mathrm{d}z} > 0$ (eq. condition using above identity)
$\implies \alpha \frac{\mathrm{d}s}{\mathrm{d}z} > 0$ (since v is always positive)
$\frac{\mathrm{d}s}{\mathrm{d}z} > 0$ (for materials with $\alpha > 0$ (most materials))

Thermodynamics Review IV

$\mathrm{d}s = \left(\frac{\partial s}{\partial T}\right)_p \mathrm{d}T + \left(\frac{\partial s}{\partial p}\right)_T \mathrm{d}p$ (general total derivative)
$\frac{\mathrm{d}s}{\mathrm{d}z} = \left(\frac{\partial s}{\partial T}\right)_p \frac{\mathrm{d}T}{\mathrm{d}z} + \left(\frac{\partial s}{\partial p}\right)_T \frac{\mathrm{d}p}{\mathrm{d}z}$ (after differentiating)
$\mathrm{d}S = \frac{mc_\mathrm{p}}{T}\,\mathrm{d}T + \left(\frac{\partial S}{\partial p}\right)_T \mathrm{d}p$ (from Thermodynamics Review II)
$\implies \left(\frac{\partial s}{\partial T}\right)_p = \frac{c_\mathrm{p}}{T}$ (in isobaric situations with $\mathrm{d}p = 0$)
$\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p$ (from Maxwell's relations)
$\left(\frac{\partial v}{\partial T}\right)_p = \alpha v$ (from Thermodynamics Review III)
$\frac{\mathrm{d}s}{\mathrm{d}z} = \frac{c_\mathrm{p}}{T}\frac{\mathrm{d}T}{\mathrm{d}z} - \left(\frac{\partial v}{\partial T}\right)_p \frac{\mathrm{d}p}{\mathrm{d}z}$ (using above identities)
$\frac{\mathrm{d}s}{\mathrm{d}z} = \frac{c_\mathrm{p}}{T}\frac{\mathrm{d}T}{\mathrm{d}z} - \alpha v \frac{\mathrm{d}p}{\mathrm{d}z}$ (using above identities)

Analyzing Equilibrium Condition III

$\frac{\mathrm{d}s}{\mathrm{d}z} > 0$ (for materials with $\alpha > 0$)
$\frac{\mathrm{d}s}{\mathrm{d}z} = \frac{c_\mathrm{p}}{T}\frac{\mathrm{d}T}{\mathrm{d}z} - \alpha v \frac{\mathrm{d}p}{\mathrm{d}z} > 0$ (from Thermodynamics Review IV)
$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g$ (in hydrostatic situations)
$\frac{c_\mathrm{p}}{T}\frac{\mathrm{d}T}{\mathrm{d}z} + \alpha g > 0$ (using above identity and $\rho v = 1$)
$\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right| < \frac{\alpha g T}{c_\mathrm{p}}$ (equilibrium condition)
$\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right| < \frac{g}{c_\mathrm{p}}$ (in ideal gases with $\alpha = 1/T$)

Bernoulli Equation

Streamlines are curves in space tangent to velocity field $\mathbf{v}(\mathbf{r}, t)$.
$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} = \frac{\mathrm{d}z}{v_z}$ (streamline equation)
Consider flow with velocity field $\mathbf{v}$ in gravitational field $\mathbf{g}$.
Simplification: assume flow is isentropic.
Simplification: assume flow is stationary (i.e. $\frac{\partial \mathbf{v}}{\partial t} = \mathbf{0}$).
$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{g} - \nabla h$ (Euler equation for isentropic flow)
$(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{g} - \nabla h$ (assuming stationary flow)
$(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\nabla v^2 - \mathbf{v}\times(\nabla\times\mathbf{v})$ (general vector identity)
Apply above vector identity and rearrange to get...
$-\mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{g} - \nabla\left(h + \frac{v^2}{2}\right)$
Consider displacement $\mathrm{d}\mathbf{l}$ along a *single* streamline, where by definition $\mathrm{d}\mathbf{l} \parallel \mathbf{v}$, and get...
$0 = \mathbf{g}\cdot\mathrm{d}\mathbf{l} - \nabla\left(h + \frac{v^2}{2}\right)\cdot\mathrm{d}\mathbf{l}$ (because $\mathbf{v}\times(\nabla\times\mathbf{v}) \perp \mathrm{d}\mathbf{l}$)
Align coordinate system so that $\mathbf{g} = -g\,\hat{\mathbf{e}}_z$.
$\implies 0 = \nabla\left(h + \frac{v^2}{2} + gz\right)\cdot\mathrm{d}\mathbf{l}$
$\implies \frac{v^2}{2} + h + g = \text{constant}$ (**Bernoulli equation**)
The above Bernoulli eq. applies only along a single streamline!

Kelvin's Circulation Theorem

Situation: consider incompressible, inviscid flow and observe the circulation of fluid elements around a closed curve C moving with the flow.
For any given closed curve C, we defined *circulation* as...
$\Gamma \equiv \oint_C \mathbf{v}\cdot\mathrm{d}\mathbf{s}$ (circulation of $\mathbf{v}$ around C)
$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0 \implies \Gamma = \text{constant}$ (**circulation theorem**)

Derivation: Circulation Theorem

Goal: prove $\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 0$ in incompressible, inviscid flow.
$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{g} - \nabla\frac{p}{\rho}$ (Euler eq. for incompressible flow)
$= -\nabla\Phi - \nabla\frac{p}{\rho}$ (in terms of gravitational potential)
$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\oint_C \mathbf{v}\cdot\mathrm{d}\mathbf{s}$ (use product rule...)
$= \oint \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\cdot\mathrm{d}\mathbf{s} + \oint \mathbf{v}\cdot\frac{\mathrm{D}(\mathrm{d}\mathbf{s})}{\mathrm{D}t}$ (split into two parts...)
$\equiv \mathrm{I} + \mathrm{II}$
$\mathrm{I} = \oint \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\cdot\mathrm{d}\mathbf{s} = -\oint \nabla\left(\Phi - \frac{p}{\rho}\right)\cdot\mathrm{d}\mathbf{s}$ (from Euler equation)
$= -\iint \nabla\times\left[\nabla\left(\Phi - \frac{p}{\rho}\right)\right]\cdot\mathrm{d}\mathbf{S}$ (Stokes' theorem)
$= 0$ ($\nabla\times(\nabla f) = \mathbf{0}$ for any scalar field f)
$\mathrm{II} = \oint \mathbf{v}\cdot\frac{\mathrm{D}(\mathrm{d}\mathbf{s})}{\mathrm{D}t} = \oint \mathbf{v}\cdot\mathrm{d}\mathbf{v}$ (using $\frac{\mathrm{D}(\mathrm{d}\mathbf{s})}{\mathrm{D}t} = \mathrm{d}\mathbf{v}$)

$= \oint \mathrm{d}\left(\frac{1}{2}|\mathbf{v}|^2\right)$ (using $\mathrm{d}(|\mathbf{v}|^2) = 2\mathbf{v}\cdot\mathrm{d}\mathbf{v}$)
$= 0$ (closed line integral of total differential is zero)
$\implies \frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \mathrm{I} + \mathrm{II} = 0 + 0 = 0$ (end of derivation)

Potential Flow

Flow with $\nabla\times\mathbf{v} = \mathbf{0}$ is called *irrotational* or *potential flow.*
Because $\nabla\times\mathbf{v} = \mathbf{0}$, potential flow may be written in terms of a velocity potential ϕ as...
$\mathbf{v} \equiv \nabla\phi$ (because $\nabla\times\mathbf{v} = \mathbf{0}$ and $\nabla\times(\nabla\phi) = \mathbf{0}$)
$\Gamma = \oint_C \mathbf{v}\cdot\mathrm{d}\mathbf{s} = 0$ (for all curves C in an irrotational $\mathbf{v}$ field)
$= \iint \nabla\times\mathbf{v}\cdot\mathrm{d}\mathbf{S}$ (proof using Stokes' theorem)
$= \iint \mathbf{0}\cdot\mathrm{d}\mathbf{S} = 0$

"Bernoulli Equation" for Potential Flow

Consider isentropic and irrotational flow.
Align coordinate system so that $\mathbf{g} = -g\,\hat{\mathbf{e}}_z$.
$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{g} - \nabla h$ (Euler eq. for isentropic flow)
$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla(h + gz)$ (for $\mathbf{g} = -g\,\hat{\mathbf{e}}_z$)
$(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\nabla v^2 - \mathbf{v}\times(\nabla\times\mathbf{v})$ (general vector identity)
$(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\nabla v^2$ (for irrotational flow)
$\frac{\partial\mathbf{v}}{\partial t} = -\nabla\left(h + \frac{v^2}{2} + gz\right)$ (using above identity)
$\mathbf{v} = \nabla\phi \implies \frac{\partial\mathbf{v}}{\partial t} = \frac{\partial(\nabla\phi)}{\partial t}$ (for potential flow)
$\implies \nabla\left(\frac{\partial\phi}{\partial t} + h + \frac{v^2}{2} + gz\right) = \mathbf{0}$ (using $\mathbf{v} = \nabla\phi$)
$\frac{\partial\phi}{\partial t} + h + \frac{v^2}{2} + gz = f(t)$ (for potential flow)
$h + \frac{v^2}{2} + gz = \text{constant}$ (for stationary potential flow)
Unlike the general Bernoulli equation in Bernoulli Equation, the above Bernoulli equation for potential flow applies to the entire flow, not just along a single streamline!

Incompressible Flow

Flow with $\rho = \text{constant}$ is called *incompressible flow.*
$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{g} - \frac{\nabla p}{\rho}$ (Euler equation in general)
$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{g} - \nabla\left(\frac{p}{\rho}\right)$ (Euler eq. for incompressible flow)
$\frac{\partial\rho}{\partial t} + \rho\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\rho = 0$ (continuity equation in general)
$\nabla\cdot\mathbf{v} = 0$ (continuity eq. for incompressible flow)

Bernoulli Equation for Incompressible Flow

$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{g} - \nabla\left(\frac{p}{\rho}\right)$ (Euler eq. for IF)
Restriction: assume flow is stationary, i.e. $\frac{\partial\mathbf{v}}{\partial t} = \mathbf{0}$.
$(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{g} - \nabla\left(\frac{p}{\rho}\right)$ (for stationary flow)
Align coordinate system so $\mathbf{g} = -g\,\hat{\mathbf{e}}_z$.
$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\left(gz + \frac{p}{\rho}\right)$ (if $\mathbf{g} = -g\,\hat{\mathbf{e}}_z$)
$(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\nabla^2 v^2 - \mathbf{v}\times(\nabla\times\mathbf{v})$ (general vector identity)
$\mathbf{v}\times(\nabla\times\mathbf{v}) = -\nabla\left(gz + \frac{p}{\rho} + \frac{v^2}{2}\right)$ (using above identity)
Consider displacement $\mathrm{d}\mathbf{l}$ along a *single* streamline, where by definition $\mathrm{d}\mathbf{l} \parallel \mathbf{v}$, and get...
$0 = \nabla\left(gz + \frac{p}{\rho} + \frac{v^2}{2}\right)\cdot\mathrm{d}\mathbf{l}$ (because $\mathbf{v}\times(\nabla\times\mathbf{v}) \perp \mathrm{d}\mathbf{l}$)
$\implies \frac{v^2}{2} + \frac{p}{\rho} + gz = \text{constant}$ (Bernoulli equation for IF)
The above Bernoulli eq. applies only along a single streamline!

Bernoulli Equation for Incompressible Potential Flow

Situation as in "**Bernoulli Equation for Incompressible Flow**"
Additionally assume flow is potential, i.e. $\nabla\times\mathbf{v} = \mathbf{0}$.
$\mathbf{v}\times(\nabla\times\mathbf{v}) = -\nabla\left(gz + \frac{p}{\rho} + \frac{v^2}{2}\right)$ (from general IF)
$\mathbf{0} = \nabla\left(gz + \frac{p}{\rho} + \frac{v^2}{2}\right)$ (for potential flow with $\nabla\times\mathbf{v} = \mathbf{0}$)
$\implies \frac{v^2}{2} + \frac{p}{\rho} + gz = \text{constant}$ (Bernoulli eq. for potential IF)
Assuming potential flow, the above Bernoulli eq. applies to the entire flow, not only along a single streamline!

Incompressible Potential Flow

Consider potential flow with velocity field $\mathbf{v}$.
$\mathbf{v} = \nabla\phi$ (in terms of velocity potential)
$\nabla\cdot\mathbf{v} = 0$ (continuity equation for incompressible flow)
Substitute $\mathbf{v} = \nabla\phi$ into continuity equation and get...
$\nabla^2\phi = 0$ (Laplace equation for incompressible potential flow)
Let $\hat{\mathbf{n}}$ be unit normal to surface containing fluid.
$\mathbf{v}\cdot\hat{\mathbf{n}} \equiv v_n = 0$ (boundary condition)
$\hat{\mathbf{n}}\cdot\nabla\phi = 0$ (using $\mathbf{v} = \nabla\phi$)

Condition for Incompressible Potential Flow

Goal: determine conditions for which flow may be approximated as incompressible, even when fluid itself is compressible.
Consider incompressible potential flow around an obstacle.
Assume obstacle moves relative to steady state flow with reference velocity $\mathbf{u}$ at position of constant reference pressure p_0.
Idea: compare change in density between a stagnation point with velocity $v = 0$ and thus maximum pressure $p_{\max}$ to density at reference point (u, p_0). If density change is small, flow is approximately incompressible.

Thermodynamics Review V

$\rho = \rho(p, S)$ (general expression for density)
$\mathrm{d}\rho = \left(\frac{\partial\rho}{\partial p}\right)_S \mathrm{d}p + \left(\frac{\partial\rho}{\partial S}\right)_p \mathrm{d}S$ (total differential)
$\mathrm{d}\rho = \left(\frac{\partial\rho}{\partial p}\right)_S \mathrm{d}p$ (for isentropic processes)
$\implies \Delta\rho = \left(\frac{\partial\rho}{\partial p}\right)_S \Delta p$ (for isentropic processes)
$\left(\frac{\partial\rho}{\partial p}\right)_S = \left(\frac{\partial(m/V)}{\partial p}\right)_S = -\frac{m}{V^2}\left(\frac{\partial V}{\partial p}\right)_S$
$\beta_{\mathrm{S}} = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S$ (definition of isentropic compressibility)
$\implies \left(\frac{\partial\rho}{\partial p}\right)_S = -\frac{m}{V^2}\left(\frac{\partial V}{\partial p}\right)_S = \frac{m}{V}\beta_{\mathrm{S}}$ (in terms of β_{S})
$= \rho\beta_{\mathrm{S}}$ (since $\rho = m/V$)
$= 1/c^2$ (since $c^2 = \frac{1}{\rho\beta_{\mathrm{S}}}$)
$\implies \Delta\rho = \frac{\Delta p}{c^2}$
c is speed of sound in material.

Finding Incompressible Flow Condition

$\frac{v^2}{2} + \frac{p}{\rho} + gz = \text{constant}$ (Bernoulli eq. for potential IF)
Simplification: assume gravitational potential is negligible.
$\frac{v^2}{2} + \frac{p}{\rho} = \text{constant}$ (neglecting gz term)
$p_{\max} - p_0 = \rho\frac{u^2}{2}$ (between reference and stagnation point)
$\Delta\rho = \frac{\Delta p}{c^2}$ (from Thermodynamics Review V)
$\rho\frac{u^2}{2} = p_{\max} - p_0 = \Delta p = c^2\Delta\rho$ (using $\Delta p = c^2\Delta\rho$)
$\frac{\Delta\rho}{\rho} = \frac{1}{2}\frac{u^2}{c^2}$ (relative fluid density change between reference and stagnation point)
Conclusion: flow is "incompressible" if $\left(\frac{u}{c}\right)^2 \ll 1$.

Incompressible Flow in Two Dimensions

Consider inviscid, incompressible flow confined to the xy plane.
$\mathbf{v} = v_x(x,y)\,\hat{\mathbf{e}}_x + v_y(x,y)\,\hat{\mathbf{e}}_y$ (velocity field in 2D)
$\nabla\cdot\mathbf{v} = 0$ (continuity equation for IF)
$\frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y}$ (from $\nabla\cdot\mathbf{v} = 0$)
$v_x \equiv \frac{\partial\psi}{\partial y}$ and $v_y \equiv -\frac{\partial\psi}{\partial x}$ (stream function)
By construction, the stream function satisfies $\frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y}$*!*

Dynamics of 2D Incompressible Flow

Consider inviscid, isentropic, and incompressible 2D flow.
$\mathbf{v} = \frac{\partial\psi}{\partial y}\,\hat{\mathbf{e}}_x - \frac{\partial\psi}{\partial x}\,\hat{\mathbf{e}}_y$ (in terms of stream function)
From Isentropic Euler Equation in Gradient-Free Form recall...
$\frac{\partial(\nabla\times\mathbf{v})}{\partial t} = \nabla\times\left[\mathbf{v}\times(\nabla\times\mathbf{v})\right]$ (Euler eq. in gradient-free form)
Plan: analyze stream function dynamics from z component of above gradient-free Euler equation.

$\nabla \times \mathbf{v} = -(0, 0, \nabla^2\psi)$ (using $\mathbf{v}$ in terms of $\psi = \psi(x, y)$)
$\mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla^2\psi(-v_y, v_x, 0)$
$\nabla \times \big[\mathbf{v} \times (\nabla \times \mathbf{v})\big]_z = \frac{\partial(v_x \nabla^2\psi)}{\partial x} + \frac{\partial(v_y \nabla^2\psi)}{\partial y}$ (z component)
$= \frac{\partial v_x}{\partial x}\nabla^2\psi + v_x\frac{\partial \nabla^2\psi}{\partial x} + \frac{\partial v_y}{\partial y}\nabla^2\psi + v_y\frac{\partial \nabla^2\psi}{\partial y}$
$= \frac{\partial^2\psi}{\partial y \partial x}\nabla^2\psi + \frac{\partial\psi}{\partial y}\frac{\partial \nabla^2\psi}{\partial x} - \frac{\partial^2\psi}{\partial x \partial y}\nabla^2\psi - \frac{\partial\psi}{\partial x}\frac{\partial \nabla^2\psi}{\partial y}$
$= \frac{\partial\psi}{\partial y}\frac{\partial \nabla^2\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial \nabla^2\psi}{\partial y}$ (eliminating common terms)
$\frac{\partial(\nabla^2\psi)}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial(\nabla^2\psi)}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial(\nabla^2\psi)}{\partial y} = 0$ (z comp. of Euler eq.)

Streamlines for 2D Incompressible Flow
$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} = \frac{\mathrm{d}z}{v_z}$ (streamline equation in 3D)
$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} \implies v_x\,\mathrm{d}y - v_y\,\mathrm{d}x = 0$ (streamline equation in 2D)
$\frac{\partial\psi}{\partial y}\,\mathrm{d}y + \frac{\partial\psi}{\partial x}\,\mathrm{d}x = \mathrm{d}\psi = 0$ (in terms of ψ)
$\mathrm{d}\psi = 0 \implies$ *ψ is constant along a streamline for 2D IF!*
Conclusion: streamlines for 2D incompressible flow are the families of curves found by setting $\psi =$ constant.

Flow Rates for 2D Incompressible Flow
In 3D, we speak of volume flow rate through a surface.
In 2D, we speak of "area flow rate" through a curve.
Consider inviscid, incompressible 2D flow with velocity field $\mathbf{v}$ and stream function ψ.
Consider a 2D curve C from $\mathbf{r}_1$ to $\mathbf{r}_1$ with tangent $\mathbf{l} = (\mathrm{d}x, \mathrm{d}y, 0)$ and thus normal $\mathbf{n} = (\mathrm{d}y, -\mathrm{d}x, 0)$.
Let $v_\perp$ denote the velocity component normal to the curve C.
$v_\perp\,\mathrm{d}l = \mathbf{v}\cdot\mathbf{n} = v_x\,\mathrm{d}y - v_y\,\mathrm{d}x$ (component normal to curve)
$= \frac{\partial\psi}{\partial y}\,\mathrm{d}y + \frac{\partial\psi}{\partial x}\,\mathrm{d}x = \mathrm{d}\psi$ (in terms of ψ)
$\mathrm{d}Q_\mathrm{A} = v_\perp\,\mathrm{d}l = \mathrm{d}\psi$ (area flow rate through segment $\mathrm{d}\mathbf{l}$)
$Q_\mathrm{A} = \int_C \mathrm{d}Q_\mathrm{A} = \int_C \mathrm{d}\psi$ (area flow through curve C)
$= \psi(\mathbf{r}_2) - \psi(\mathbf{r}_1)$ (area flow through curve C)
$Q_\mathrm{m} = \int_C \rho\,\mathrm{d}Q_\mathrm{A} = \rho\int \mathrm{d}Q_\mathrm{A}$ (for incompressible flow)
$= \rho\big[\psi(\mathbf{r}_1) - \psi(\mathbf{r}_1)\big]$ (mass flow through curve C)

Potential and Incompressible 2D Flow
Consider inviscid, incompressible, potential flow in 2D.
$\mathbf{v} = v_x(x, y)\,\hat{\mathbf{e}}_x + v_y(x, y)\,\hat{\mathbf{e}}_y$ (for 2D flow)
$v_x = \frac{\partial\psi}{\partial y}$ and $v_y = -\frac{\partial\psi}{\partial x}$ (for incompressible 2D flow)
$\mathbf{v} = \nabla\phi$ (for potential flow)
$v_x = \frac{\partial\phi}{\partial x}, \quad v_y = \frac{\partial\phi}{\partial y}$ (from $\mathbf{v} = \nabla\phi$ in 2D)
$v_x = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$ (combining ϕ and ψ formulations)
$v_y = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$ (combining ϕ and ψ formulations)
$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$ and $\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$ form a set of Cauchy-Riemann eqs. for the functions ϕ and ψ, motivating the introduction of...
$w(x, y) \equiv \phi(x, y) + i\psi(x, y)$ (complex velocity potential)
$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{d}w}{\mathrm{d}x} = v_x - iv_y = \frac{1}{i}\frac{\mathrm{d}w}{\mathrm{d}y}$ (complex derivative)

TODO: check out vaje derivation for derivatives.

Drag Around Obstacles in Potential Flow
Consider inviscid, incompressible, and potential flow around an obstacle. Assume obstacle moves with reference velocity $\mathbf{u}$ relative to distant, homogeneous steady state flow.
Model: net flow around obstacle is the sum of homogeneous velocity $\mathbf{u}$ and a velocity dipole centered on obstacle with source at back of obstacle and sink in front of obstacle.
(Front/back determined with respect to direction of $\mathbf{u}$).
Goal: determine velocity field $\mathbf{v}$ around obstacle.

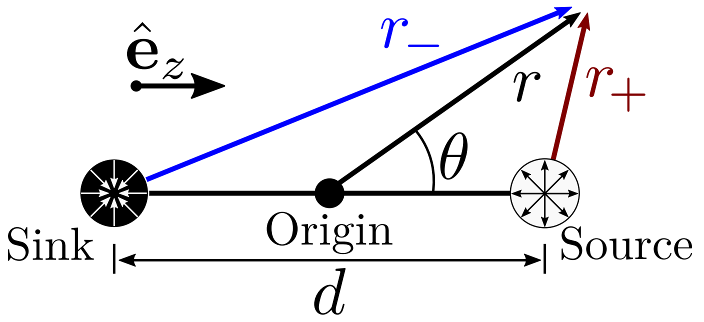

Figure 4: Geometry of a velocity dipole.

Finding Velocity Field Around Obstacle
$\mathbf{v} = \nabla\phi$ (for potential flow)
$\nabla\cdot\mathbf{v} = 0$ (continuity equation for incompressible flow)
$\nabla^2\phi = 0$ (Laplace equation for potential IF)
Place origin at midpoint between sink and source.
Align coordinate system to z axis points from sink to source.
$\phi_\mathrm{dipole} = -\frac{a}{r_+} + \frac{a}{r_-}$ (dipole field ansatz)
a is a constant encoding dipole strength.
Let d denote distance between source and sink.
Let $\mathbf{d} = d\,\hat{\mathbf{e}}_z$ denote vector distance from source to sink.
r_+ is distance from source to position vector $\mathbf{r}$.
r_- is distance from sink to position vector $\mathbf{r}$.
$r_+^2 = r^2 - rd\cos\theta + \left(\frac{d}{2}\right)^2$ (law of cosines)
$r_-^2 = r^2 + rd\cos\theta + \left(\frac{d}{2}\right)^2$ (law of cosines)
$\phi_\mathrm{dipole} = -\frac{a}{r_+} + \frac{a}{r_-}$ (in general)
$\approx -\frac{a}{r}\left(1 + \frac{d\cos\theta}{2r}\right) + \frac{a}{r}\left(1 - \frac{d\cos\theta}{2r}\right)$ (up to $\mathcal{O}(d/r)$)
$= -\frac{ad\cos\theta}{r^2}$
Let $\mathbf{a} = ad\,\hat{\mathbf{e}}_z = a\mathbf{d}$ **TODO:** a and $\mathbf{a}$ have different units
$\phi_\mathrm{dipole} = -\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}$ (in terms of $\mathbf{a}$)
$\mathbf{v}_\mathrm{dipole} = \nabla\phi_\mathrm{dipole}$ (dipole velocity field)
$= -\nabla\frac{\mathbf{a}\cdot\mathbf{r}}{r^3} = \frac{\mathbf{a}}{r^3} + 3\frac{\mathbf{a}\cdot\mathbf{r}}{r^4}\,\hat{\mathbf{r}}$ (from above ϕ_dipole)
$= \frac{3(\mathbf{a}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{a}}{r^3}$

Kinetic Energy of Dipole Flow
Goal: compute kinetic energy E of dipole flow around obstacle.
Notation: for shorthand, let $\phi_\mathrm{dipole} \to \phi$ and $\mathbf{v}_\mathrm{dipole} \to \mathbf{v}$.
$E = \frac{1}{2}\iiint \rho v^2\,\mathrm{d}V$ (from first principles)
$= \frac{\rho}{2}\iiint v^2\,\mathrm{d}V$ (for incompressible flow)
$= \frac{\rho}{2}\iiint \big[u^2 + (v^2 - u^2)\big]\,\mathrm{d}V$ (decomposition)
$= \frac{\rho}{2}\iiint u^2\,\mathrm{d}V + \frac{\rho}{2}\iiint(\mathbf{v} - \mathbf{u})\cdot(\mathbf{v} + \mathbf{u})\,\mathrm{d}V$ (decomposition)
Let integral run over the closed region bounded on the inner surface by the obstacle of volume V_0 and on the outer surface by a sphere of volume $V \gg V_0$.
Let $\mathrm{I} \equiv \frac{\rho}{2}\iiint u^2\,\mathrm{d}V$ (first integral).
Let $\mathrm{II} \equiv \frac{\rho}{2}\iiint(\mathbf{v} + \mathbf{u})\cdot(\mathbf{v} - \mathbf{u})\,\mathrm{d}V$ (second integral).
$\mathrm{I} = \frac{\rho u^2}{2}(V - V_0)$ (derived below)
$\mathrm{II} = 2\pi\rho(\mathbf{a}\cdot\mathbf{u}) - \frac{\rho u^2}{2}V$ (derived below)
$E = 2\pi\rho(\mathbf{a}\cdot\mathbf{u}) - \frac{\rho u^2}{2}V_0$ (**kinetic energy of dipole flow**)

Derivation: Kinetic Energy of Dipole Flow
Evaluating First Integral
$\mathrm{I} = \frac{\rho}{2}\iiint u^2\,\mathrm{d}V = \frac{\rho u^2}{2}\iiint \mathrm{d}V$ (because u is constant)
$= \frac{\rho u^2}{2}(V - V_0)$

Evaluating Second Integral
$\mathrm{II} = \frac{\rho}{2}\iiint(\mathbf{v} + \mathbf{u})\cdot(\mathbf{v} - \mathbf{u})\,\mathrm{d}V$
$\mathbf{v} = \nabla\phi$ (for potential flow)
$\nabla\mathbf{u}\cdot\mathbf{r} = (\nabla\mathbf{r})\cdot\mathbf{u} + (\nabla\mathbf{u})\cdot\mathbf{r} = \mathbf{u}$ (because $\mathbf{u}$ is constant)
$\implies \mathbf{v} + \mathbf{u} = \nabla(\phi + \mathbf{u}\cdot\mathbf{r})$
$(\mathbf{v} + \mathbf{u})\cdot(\mathbf{v} - \mathbf{u}) = \big[\nabla(\phi + \mathbf{u}\cdot\mathbf{r})\big]\cdot(\mathbf{v} - \mathbf{u})$
$= \nabla\cdot\big[(\phi + \mathbf{u}\cdot\mathbf{r})(\mathbf{v} - \mathbf{u})\big]$ (aux. identity)

Derivation: $\nabla \cdot \big[(\phi + \mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\big] = \big[\nabla(\phi + \mathbf{u}\cdot\mathbf{r})\cdot(\mathbf{v}-\mathbf{u})\big] + (\phi + \mathbf{u}\cdot\mathbf{r})\big[\nabla\cdot(\mathbf{v}-\mathbf{u})\big]$, then apply $\nabla\cdot\mathbf{v} = 0$ (for incompressible fluids) and $\nabla\cdot\mathbf{u} = 0$ (constant reference velocity).

$\mathrm{II} = \frac{\rho}{2}\iiint(\mathbf{v}+\mathbf{u})\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}V$
$\quad = \frac{\rho}{2}\iiint\nabla\cdot\big[(\phi+\mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\big]\,\mathrm{d}V$ (above identity)
$\quad = \frac{\rho}{2}\oiint(\phi+\mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\cdot\mathrm{d}\mathbf{S}$ (divergence theorem)

Next: divide closed surface integral into integrals over inner surface (around obstacle) and outer surface (over large sphere).
$\mathrm{II}_{\mathrm{in}}$ is integral over inner surface.
$\mathrm{II}_{\mathrm{out}}$ is integral over outer surface.

Second Integral Part 1: Inner Surface

Let $\hat{\mathbf{n}}$ denote outward normal to inner integration surface (and thus inward normal to obstacle).
Let $u_\perp$ and $v_\perp$ denote components normal to inner surface.
Boundary condition: $v_\perp - u_\perp = 0$, i.e. relative fluid velocity normal to obstacle/inner surface is zero.

$v_\perp - u_\perp = (\mathbf{v}-\mathbf{u})\cdot\hat{\mathbf{n}} = 0$ (in vector form)
$\mathrm{II}_{\mathrm{in}} = \frac{\rho}{2}\iint_{\mathrm{in}}(\phi+\mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\cdot\mathrm{d}\mathbf{S}$
$\quad = \frac{\rho}{2}\iint_{\mathrm{in}}(\phi+\mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ (using $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S$)
$\quad = \frac{\rho}{2}\iint_{\mathrm{in}}0\,\mathrm{d}S = 0$ (using BC $(\mathbf{v}-\mathbf{u})\cdot\hat{\mathbf{n}} = 0$)
$\implies \mathrm{II} = \mathrm{II}_{\mathrm{out}} - \mathrm{II}_{\mathrm{in}} = \mathrm{II}_{\mathrm{out}}$ (since $\mathrm{II}_{\mathrm{in}} = 0$)

Second Integral Part 2: Outer Surface

Outer integration surface is a large sphere of radius R.
$\mathrm{d}\mathbf{S} = R^2\,\mathrm{d}\Omega\,\hat{\mathbf{r}}$ (vector surface element of outer surface)
$\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ is element of solid angle
$\mathrm{II}_{\mathrm{out}} = \frac{\rho}{2}\iint_{\mathrm{out}}(\phi+\mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\cdot\mathrm{d}\mathbf{S}$
$\quad = \frac{\rho}{2}\iint_{\mathrm{out}}(\phi+\mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\cdot\hat{\mathbf{r}}R^2\,\mathrm{d}\Omega$
$\phi = -\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}$ (dipole velocity potential)
$\mathbf{v} = \frac{3(\mathbf{a}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{a}}{r^3}$ (dipole velocity field)
Substitute ϕ and $\mathbf{v}$ into $\mathrm{II}_{\mathrm{out}}$ and multiply out to get...
$\mathrm{II}_{\mathrm{out}} = \frac{\rho}{2}\iint_{\mathrm{out}}\Big[-(\mathbf{u}\cdot\hat{\mathbf{r}})^2 r + \frac{3(\mathbf{a}\cdot\hat{\mathbf{r}})(\mathbf{u}\cdot\hat{\mathbf{r}})}{r^2} - \frac{2(\mathbf{a}\cdot\hat{\mathbf{r}})^2}{r^5}\Big]R^2\,\mathrm{d}\Omega$
Evaluate at $r = R$ (i.e. on outer integration surface) and get...
$\mathrm{II}_{\mathrm{out}} = \frac{\rho}{2}\iint_{\mathrm{out}}\Big[-(\mathbf{u}\cdot\hat{\mathbf{r}})^2R^3 + 3(\mathbf{a}\cdot\hat{\mathbf{r}})(\mathbf{u}\cdot\hat{\mathbf{r}}) - \frac{2(\mathbf{a}\cdot\hat{\mathbf{r}})^2}{R^3}\Big]\,\mathrm{d}\Omega$
Neglect $\sim R^{-3}$ term (which vanishes for large R) and get...
$\mathrm{II}_{\mathrm{out}} = \frac{\rho}{2}\iint_{\mathrm{out}}\big[-(\mathbf{u}\cdot\hat{\mathbf{r}})(\mathbf{u}\cdot\hat{\mathbf{r}})R^3 + 3(\mathbf{a}\cdot\hat{\mathbf{r}})(\mathbf{u}\cdot\hat{\mathbf{r}})\big]\,\mathrm{d}\Omega$
Let $\mathbf{A}\equiv-\mathbf{u}$, $\mathbf{B}\equiv R^3\mathbf{u}$ and $\mathbf{C}\equiv3\mathbf{a}$, $\mathbf{D}\equiv\mathbf{u}$
$\implies \mathrm{II}_{\mathrm{out}} = \frac{\rho}{2}\iint_{\mathrm{out}}\big[(\mathbf{A}\cdot\hat{\mathbf{r}})(\mathbf{B}\cdot\hat{\mathbf{r}}) + (\mathbf{C}\cdot\hat{\mathbf{r}})(\mathbf{D}\cdot\hat{\mathbf{r}})\big]\,\mathrm{d}\Omega$
Idea: write above integrals are averages over solid angle.
$\mathrm{II}_{\mathrm{out}} = 4\pi\cdot\frac{\rho}{2}\Big[\overline{(\mathbf{A}\cdot\hat{\mathbf{r}})(\mathbf{B}\cdot\hat{\mathbf{r}})} + \overline{(\mathbf{C}\cdot\hat{\mathbf{r}})(\mathbf{D}\cdot\hat{\mathbf{r}})}\Big]$ (avg. over Ω)
Notation change: let $\hat{\mathbf{r}}\to\hat{\mathbf{n}}$ and $(\hat{\mathbf{r}})_i\to n_i$.
$\mathrm{II}_{\mathrm{out}} = 2\pi\rho\Big(\overline{A_in_iB_jn_j} + \overline{C_in_iD_jn_j}\Big)$ (using index notation)
Because $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are constants, we move them out of the average to get...
$\mathrm{II}_{\mathrm{out}} = 2\pi\rho\Big(A_iB_j\overline{n_in_j} + C_iD_j\overline{n_in_j}\Big)$ (since **A-D** are constant)
$\quad = 2\pi\rho\big(A_iB_j + C_iD_j\big)\frac{\delta_{ij}}{3}$ (using $\overline{n_in_j} = \delta_{ij}/3$)
$\quad = \frac{2\pi\rho}{3}\big(\mathbf{A}\cdot\mathbf{B} + \mathbf{C}\cdot\mathbf{D}\big)$ (returning to vector notation)
$\quad = \frac{2\pi\rho}{3}\big(-u^2R^3 + 3\mathbf{a}\cdot\mathbf{u}\big)$ (in terms of **A-D**)
$\mathrm{II} = \mathrm{II}_{\mathrm{out}} = 2\pi\rho(\mathbf{a}\cdot\mathbf{u}) - \frac{2\pi\rho}{3}u^2R^3$ (since $\mathrm{II}_{\mathrm{in}} = 0$)
$\quad = 2\pi\rho(\mathbf{a}\cdot\mathbf{u}) - \frac{\rho u^2}{2}V$ (using $V = 4\pi R^3/3$)

Result: Kinetic Energy of Dipole Flow

$E = \frac{\rho}{2}\iiint\big[u^2 + (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}+\mathbf{u})\big]\,\mathrm{d}V$ (for review)
$\quad \equiv \mathrm{I} + \mathrm{II}$ (for review)
$$= \underbrace{\frac{\rho u^2}{2}(V - V_0)}_{\mathrm{I}} + \underbrace{\Big[2\pi\rho(\mathbf{a}\cdot\mathbf{u}) - \frac{\rho u^2}{2}V\Big]}_{\mathrm{II}}$$
$\quad = 2\pi\rho(\mathbf{a}\cdot\mathbf{u}) - \frac{\rho u^2}{2}V_0$ (**kinetic energy of dipole flow**)

Drag Forces and D'Alembert's Paradox

Consider inviscid, incompressible, and potential flow around an obstacle of volume V_0. Assume obstacle moves with reference velocity $\mathbf{u}$ relative to distant, homogeneous steady state flow.
Let $\mathbf{a}$ encode the strength and direction of the velocity dipole field surrounding the obstacle.
$E = 2\pi\rho(\mathbf{a}\cdot\mathbf{u}) - \frac{\rho u^2}{2}V_0$ (kinetic energy of dipole flow)
Assumption: assume $\mathbf{a}\propto\mathbf{u}$, i.e. strength of dipole flow around obstacle is proportional to relative velocity between obstacle and steady-state flow.
$\implies \mathbf{a}\cdot\mathbf{u}\sim u^2$ (assuming $\mathbf{a}\propto\mathbf{u}$)
$\implies E\sim u^2$, which motivates the introduction of...
$E \equiv \frac{1}{2}m_{ij}u_iu_j$ (where m is a *induced mass tensor*)

D'Alembert's Paradox

Situation as in Drag Around Obstacles in Potential Flow.
$E = \frac{1}{2}m_{ik}u_iu_j$ (kinetic energy of dipole flow)
Let $\mathbf{F}$ denote external force acting on obstacle.
Acceleration of obstacle through fluid increases fluid momentum.
$\mathrm{d}\mathbf{p} = \mathbf{F}\,\mathrm{d}t$ (change in fluid momentum in time $\mathrm{d}t$)
$\mathrm{d}W = \mathbf{F}\cdot\mathrm{d}\mathbf{s} = \mathbf{F}\cdot\mathbf{u}\,\mathrm{d}t = \mathbf{u}\cdot\mathrm{d}\mathbf{p}$ (work of force on fluid)
By the work-energy theorem, change in fluid kinetic energy is...
$\mathrm{d}E = \mathrm{d}W = \mathbf{u}\cdot\mathrm{d}\mathbf{p} = u_i\,\mathrm{d}p_i$ (change in fluid energy)
Then compare $\mathrm{d}E$ to $E = \frac{1}{2}m_{ij}u_iu_j$ to get...
$p_i = m_{ij}u_j$ (from comparison to above E)
$E = 2\pi\rho(\mathbf{a}\cdot\mathbf{u}) - \frac{\rho u^2}{2}V_0$ (derived in previous section)
$\quad = \frac{1}{2}(4\pi\rho a_i - \rho u_iV_0)u_i$ (by components)
Combine with $\mathrm{d}E = u_i\,\mathrm{d}p_i$ and $p_i = m_{ij}u_j$ to conclude...
$p_i = 4\pi\rho a_i - \rho u_iV_0$ (by components)
$\mathbf{p} = 4\pi\rho\mathbf{a} - \rho V_0\mathbf{u}$ (in vector form)
$\dot{\mathbf{p}} = 4\pi\rho\dot{\mathbf{a}} - \rho V_0\dot{\mathbf{u}}$
Reaction force of fluid on body is...
$\mathbf{F}_{\mathrm{react}} = -\dot{\mathbf{p}} = -4\pi\rho\dot{\mathbf{a}} + \rho V_0\dot{\mathbf{u}}$
Component of $\mathbf{F}_{\mathrm{react}}$ parallel to $\mathbf{u}$ is *drag*.
Component of $\mathbf{F}_{\mathrm{react}}$ perpendicular to $\mathbf{u}$ is *buoyancy*.

D'Alembert's Paradox

Assuming $\mathbf{a}\propto\mathbf{u}$, drag force on body is zero if object velocity $\mathbf{u}$ relative to fluid is constant. Experiment shows that even objects moving through fluid with constant velocity experience drag.
Resolution: no real fluids are perfectly ideal and potential.

Viscous Fluids

Viscous fluids exhibit energy dissipation during fluid motion due shear stresses between relatively-moving fluid particles.
Viscous fluid flow is thermodynamically irreversible.
Concept: energy dissipation is a result of thermodynamic irreversibility of viscous fluid motion.

Deriving the Navier-Stokes Equation

Plan: generalize the Euler equation to an equation of motion that applies to viscous fluids.

Reinterpreting the Euler Equation

First recall the Euler equation for an *ideal* fluid...
$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{g} - \frac{\nabla p}{\rho}$ (Euler equation)
Assume fluid is isotropic; from Stress Tensors for Common Deformations recall...
$p_{ij} = -p\delta_{ij}$ (stress tensor for isotropic loads)
Write ∇p in terms of isotropic stress tensor $p_{ij} = -p\delta_{ij}$ as...
$(\nabla p)_i = \frac{\partial p}{\partial x_i} = \frac{\partial(p\delta_{ij})}{\partial x_j} = -\frac{\partial p_{ij}}{\partial x_j}$ (in terms of p_{ij})
$\frac{\mathrm{D}v_i}{\mathrm{D}t} = g_i + \frac{1}{\rho}\frac{\partial p_{ij}}{\partial x_j}$ (Euler equation in terms of p_{ij})
$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{g} + \frac{1}{\rho}\nabla\cdot\mathsf{p}$ (in vector form)
Idea: generalize Euler equation to apply to viscous fluids by

including non-isotropic tensor terms representing viscous forces.

Viscous Stress Tensor

Goal: determine a stress tensor p'_{ij} to represent viscous forces. We make the following heuristic considerations...

(i) p'_{ij} should depend on velocity gradient $\frac{\partial v_i}{\partial x_j}$ (because viscosity arises from relative motion of fluid particles).

(ii) We expect $p'_{ij} \to 0$ as $\frac{\partial v_i}{\partial x_j} \to 0$, which suggests linear proportionality $p'_{ij} \propto \frac{\partial v_i}{\partial x_j}$.

(iii) We expect $p'_{ij} = 0$ for a rigidly-rotating fluid mass (no relative motion of fluid particles) which suggests the *symmetrized* construction $p'_{ij} \propto \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$, which recovers $p'_{ij} = 0$ for rigid fluid rotation.

(iv) A relationship between stress and velocity gradient in isotropic fluids exhibiting zero viscous stress during rigid rotations can have only two independent (scalar) parameters, e.g. η and ζ.

A general rank-two tensor satisfying these conditions is...

$p'_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right) + \zeta\frac{\partial v_k}{\partial x_k}\delta_{ij}$ **(visc. stress tensor)**

$\mathsf{p}' = \eta\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^\top\right] - \left(\frac{2}{3}\eta + \zeta\right)(\nabla\cdot\mathbf{v})\mathsf{I}$ (vector form)

η is fluid's *dynamic viscosity*.

ζ is fluid's *bulk viscosity*.

I is identity (unit) tensor.

$\nabla\cdot\mathbf{v} = \frac{\partial v_k}{\partial x_k} = 0$ (for incompressible flow)

$\implies p'_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ (for incompressible flow)

$\implies \mathsf{p}' = \eta\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^\top\right]$ (for incompressible flow)

Divergence of the Viscous Stress Tensor

Motivation: find divergence $\frac{\partial p'_{ij}}{\partial x_j}$ of viscous stress tensor, which is used to generalize Euler equation to viscous fluids.

$p'_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right) + \zeta\frac{\partial v_k}{\partial x_k}\delta_{ij}$ (visc. stress tensor)

$$
\begin{aligned}
\frac{\partial p'_{ij}}{\partial x_j} &= \eta\left(\frac{\partial^2 v_i}{\partial x_j^2} + \frac{\partial^2 v_j}{\partial x_j \partial x_i} - \frac{2}{3}\frac{\partial}{\partial x_j}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right) + \zeta\frac{\partial}{\partial x_j}\frac{\partial v_k}{\partial x_k}\delta_{ij}\\
&= \eta\left(\frac{\partial^2 v_i}{\partial x_j^2} + \frac{\partial^2 v_j}{\partial x_j \partial x_i} - \frac{2}{3}\frac{\partial^2 v_k}{\partial x_i \partial x_k}\right) + \zeta\frac{\partial^2 v_k}{\partial x_i \partial x_k} && \text{(using up } \delta_{ij})\\
&= \eta\left(\frac{\partial^2 v_i}{\partial x_j^2} + \frac{\partial^2 v_j}{\partial x_j \partial x_i} - \frac{2}{3}\frac{\partial^2 v_j}{\partial x_i \partial x_j}\right) + \zeta\frac{\partial^2 v_j}{\partial x_i \partial x_j} && (k \to j)\\
&= \eta\frac{\partial^2 v_i}{\partial x_j^2} + \frac{\partial^2 v_j}{\partial x_i \partial x_j}\left(\frac{\eta}{3} + \zeta\right) && \text{(combining like terms)}
\end{aligned}
$$

$\nabla\cdot\mathsf{p}' = \eta\nabla^2\mathbf{v} + \nabla(\nabla\cdot\mathbf{v})\cdot\left(\frac{\eta}{3} + \zeta\right)$ (in vector form)

The Navier-Stokes Equation

$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{g} + \frac{1}{\rho}\nabla\cdot\mathsf{p} = \mathbf{g} - \frac{\nabla p}{\rho}$ (Euler equation)

We then add on $\nabla\cdot\mathsf{p}'$ to generalize the Euler equation to viscous flow. The result is...

$$
\begin{aligned}
\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} &= \mathbf{g} - \frac{\nabla p}{\rho} + \frac{1}{\rho}\nabla\cdot\mathsf{p}' && \textbf{(Navier-Stokes equation)}\\
&= \mathbf{g} - \frac{\nabla p}{\rho} + \frac{1}{\rho}\left[\eta\nabla^2\mathbf{v} + \left(\frac{\eta}{3} + \zeta\right)\nabla(\nabla\cdot\mathbf{v})\right]
\end{aligned}
$$

$\nabla\cdot\mathbf{v} = 0$ (for incompressible flow)

$(\nabla p)/\rho = \nabla(p/\rho)$ (for incompressible flow)

$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{g} - \nabla\frac{p}{\rho} + \frac{\eta}{\rho}\nabla^2\mathbf{v}$ **(NS eq. for incompressible flow)**

Kinematic Viscosity and Pressure Field

$\nu \equiv \eta/\rho$ (definition of kinematic viscosity)

Motivation: kinematic viscosity simplifies the NS equation.

Consider incompressible flow in negligible gravitational field.

$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\frac{p}{\rho} + \frac{\eta}{\rho}\nabla^2\mathbf{v}$ (NS equation for IF)

$(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\nabla v^2 - \mathbf{v}\times(\nabla\times\mathbf{v})$ (general vector identity)

$\frac{\partial\mathbf{v}}{\partial t} - \mathbf{v}\times(\nabla\times\mathbf{v}) = -\nabla\left(\frac{p}{\rho} + \frac{v^2}{2}\right) + \frac{\eta}{\rho}\nabla^2\mathbf{v}$ (from identity)

Take curl of above equation, use $\nabla\times(\nabla\phi) = \mathbf{0}$ for any scalar field ϕ, and introduce $\nu \equiv \eta/\rho$ to get...

$\frac{\partial(\nabla\times\mathbf{v})}{\partial t} = \nabla\times\left[\mathbf{v}\times(\nabla\times\mathbf{v})\right] + \nu\nabla\times(\nabla^2\mathbf{v})$

Pressure Field

Consider incompressible flow in negligible gravitational field.

Motivation: find fluid pressure p from known velocity field $\mathbf{v}$.

$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\frac{p}{\rho} + \frac{\eta}{\rho}\nabla^2\mathbf{v}$ (NS equation for IF)

$\nabla\cdot\mathbf{v} = 0$ (for incompressible fluids)

$\implies \nabla\cdot(\nabla^2\mathbf{v}) = \nabla^2(\nabla\cdot\mathbf{v}) = 0$ (for incompressible fluids)

Take divergence of above Navier-Stokes equation to get...

$\frac{\partial(\nabla\cdot\mathbf{v})}{\partial t} + \nabla\cdot\left[(\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla\cdot\left(\nabla\frac{p}{\rho}\right) + \frac{\eta}{\rho}\nabla\cdot(\nabla^2\mathbf{v})$

...which on $\nabla\cdot\mathbf{v} = 0$ and $\nabla\cdot(\nabla^2\mathbf{v}) = 0$ simplifies to

$\nabla\cdot\left[(\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla\cdot\left(\nabla\frac{p}{\rho}\right)$

$$
\begin{aligned}
\left\{\nabla\cdot\left[(\mathbf{v}\cdot\nabla)\mathbf{v}\right]\right\}_i &= \frac{\partial}{\partial x_i}v_j\frac{\partial v_i}{\partial x_j} = v_j\frac{\partial}{\partial x_i}\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j}\\
&= v_j\frac{\partial}{\partial x_j}\frac{\partial v_i}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} && \text{(rearranging deriv.)}\\
&= \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} && \left(\frac{\partial v_i}{\partial x_i} = \nabla\cdot\mathbf{v} = 0 \text{ for IF}\right)
\end{aligned}
$$

$\implies \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2}$ (equation for pressure field)

Energy Dissipation in Incompressible Flow

Consider incompressible flow in a negligible gravitational field around an obstacle moving with velocity $\mathbf{u}$ relative to the flow far from the obstacle.

$$
\begin{aligned}
E &= \frac{1}{2}\iiint \rho v^2\,\mathrm{d}V && \text{(fluid's kinetic energy)}\\
&= \frac{\rho}{2}\iiint v^2\,\mathrm{d}V && \text{(for incompressible fluids)}
\end{aligned}
$$

$\frac{\mathrm{d}E}{\mathrm{d}t} = \rho\iiint \mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t}\,\mathrm{d}V$

Some Auxiliary Vector Identities

To be used and referenced below in Analysis of Dissipation

$\mathbf{v}\cdot\left[(\mathbf{v}\cdot\nabla)\mathbf{v}\right] + \mathbf{v}\cdot\nabla p = (\mathbf{v}\cdot\nabla)\left(\frac{\rho v^2}{2} + p\right)$ (I)

$\nabla\cdot(\mathbf{v}\cdot\mathsf{p}') = (\nabla\mathbf{v})\cdot\mathsf{p}' + \mathbf{v}\cdot(\nabla\cdot\mathsf{p}')$ (II)

$$
\begin{aligned}
\nabla\cdot(\mathbf{v}f) &= f(\nabla\cdot\mathbf{v}) + \mathbf{v}\cdot\nabla f && \text{(in general for a scalar field } f)\\
&= (\mathbf{v}\cdot\nabla)f && \text{(for IF } \nabla\cdot\mathbf{v} = 0;\ \text{III)}
\end{aligned}
$$

Analysis of Dissipation

Goal: find rate of change of fluid's kinetic energy.

$\frac{\mathrm{d}E}{\mathrm{d}t} = \rho\iiint \mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t}\,\mathrm{d}V$ (from above)

$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot\mathsf{p}'$ (NS equation)

$\frac{\partial v_i}{\partial t} = -v_j\frac{\partial v_i}{\partial x_j} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial p'_{ij}}{\partial x_j}$ (by components)

$\rho v_i\frac{\partial v_i}{\partial t} = -\rho v_i v_j\frac{\partial v_i}{\partial x_j} - v_i\frac{\partial p}{\partial x_i} + v_i\frac{\partial p'_{ij}}{\partial x_j}$ (by components)

$$
\begin{aligned}
\rho\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t} &= -\rho\mathbf{v}\cdot\left[(\mathbf{v}\cdot\nabla)\mathbf{v}\right] - \mathbf{v}\cdot\nabla p + \mathbf{v}\cdot(\nabla\cdot\mathsf{p}') && \text{(vector form)}\\
&= -(\mathbf{v}\cdot\nabla)\left(\frac{\rho v^2}{2} + p\right) + \mathbf{v}\cdot(\nabla\cdot\mathsf{p}') && \text{(from I)}\\
&= -(\mathbf{v}\cdot\nabla)\left(\frac{\rho v^2}{2} + p\right) + \nabla\cdot(\mathbf{v}\cdot\mathsf{p}') - (\nabla\mathbf{v})\cdot\mathsf{p}' && \text{(from II)}\\
&= -\nabla\cdot\left[\mathbf{v}\left(\frac{\rho v^2}{2} + p\right)\right] + \nabla\cdot(\mathbf{v}\cdot\mathsf{p}') - (\nabla\mathbf{v})\cdot\mathsf{p}' && \text{(III)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \rho\iiint \mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t}\,\mathrm{d}V && \text{(for review)}\\
&= \iiint -\nabla\cdot\left[\mathbf{v}\left(\frac{\rho v^2}{2} + p\right)\right] + \nabla\cdot(\mathbf{v}\cdot\mathsf{p}')\,\mathrm{d}V\\
&\quad - \iiint(\nabla\mathbf{v})\cdot\mathsf{p}'\,\mathrm{d}V && \text{(using above results)}\\
&= \oiint\left[(\mathbf{v}\cdot\mathsf{p}') - \mathbf{v}\left(\frac{\rho v^2}{2} + p\right)\right]\cdot\mathrm{d}\mathbf{S} && \text{(divergence theorem)}\\
&\quad - \iiint(\nabla\mathbf{v})\cdot\mathsf{p}'\,\mathrm{d}V
\end{aligned}
$$

Interpreting Results

Split integral into three parts...

$\oiint\left(-\frac{\rho v^2}{2}\mathbf{v}\right)\cdot\mathrm{d}\mathbf{S}$ (transport of kinetic energy)

$\oiint(\mathbf{v}\cdot\mathsf{p}' - p\mathbf{v})\cdot\mathrm{d}\mathbf{S}$ (power of pressure and viscous forces)

$\iiint\left[-(\nabla\mathbf{v})\cdot\mathsf{p}'\right]\mathrm{d}V$ (dissipation in fluid volume)

$$
\begin{aligned}
P_{\text{dis}} &= -\iiint(\nabla\mathbf{v})\cdot\mathsf{p}'\,\mathrm{d}V && \text{(dissipation term)}\\
&= -\iiint\frac{\partial v_i}{\partial x_j}p'_{ij}\,\mathrm{d}V && \text{(in component form)}\\
&= -\frac{1}{2}\iiint\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)p'_{ij}\,\mathrm{d}V && \text{(symmetrizing; } p_{ij} = p_{ji})
\end{aligned}
$$

From Viscous Stress Tensor recall...

$p'_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$ (viscous stress tensor for IF)

$\implies P_{\text{dis}} = -\frac{\eta}{2} \iiint \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 \mathrm{d}V$

Additional result: P_{dis} must be negative on physical grounds (dissipation decreases fluid kinetic energy) and $\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2$ is always positive. Conclusion: *η must be positive on physical grounds!*

Reynolds Number

The Reynolds number is a dimensionless parameter used to characterize flows.

Law of Similarity

Consider stationary, incompressible flow around an obstacle with characteristic dimension l in a negligible gravitational field.
Goal: analyze the flow in dimensionless form.
The flow is characterized by the following three quantities...

(i) relative velocity u between object and steady flow,

(ii) linear dimension l of obstacle, and

(iii) kinematic viscosity ν of flow.

These three parameters have one dimensionless combination:
$\text{Re} \equiv \frac{ul}{\nu} = \frac{ul\eta}{\rho}$ (**definition of Reynold's number**)
$\implies$ On dimensional grounds, the velocity and pressure fields for the above flow may be written in the general form...
$\mathbf{v} = u\mathbf{f}(\mathbf{r}/l, \text{Re})$ (where $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$)
$p = \rho u^2 f\left(\frac{\mathbf{r}}{l}, \text{Re}\right)$ (where $f : \mathbb{R}^3 \to \mathbb{R}$)
$\implies$ **Law of similarity**: flows around objects of different sizes but equal proportions by fluids with different viscosities are related by rescaling the dimensionless coordinate $\mathbf{r}/l$ and $\mathbf{v}/u$ as long as the flows' Reynold's numbers are equal.

Flow with Small Reynolds Numbers

Consider stationary, incompressible flow around an obstacle with characteristic dimension l in a negligible gravitational field.
Let u denote relative velocity between object and steady flow.
$\text{Re} = \frac{ul}{\nu} = \frac{\rho ul}{\eta}$ (flow's Reynolds number)
$(\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v}$ (stationary NS eq.)
Goal: simplify the NS equation for small Reynolds numbers.
$(\mathbf{v} \cdot \nabla)\mathbf{v} \sim \frac{u^2}{l}$ (up to order of magnitude)
$\nu\nabla^2\mathbf{v} \sim \nu\frac{u}{l^2}$ (up to order of magnitude)
$(\mathbf{v} \cdot \nabla)\mathbf{v}/(\nu\nabla^2\mathbf{v}) \sim \frac{ul}{\nu} = \text{Re}$
$\implies (\mathbf{v} \cdot \nabla)\mathbf{v} \ll \nu\nabla^2\mathbf{v}$ for $\text{Re} \ll 1$
Conclusion: for $\text{Re} \ll 1$, the Navier-Stokes equation for stationary, incompressible flow simplifies to...
$\nu\nabla^2\mathbf{v} = \frac{1}{\rho}\nabla p$ (for $\text{Re} \ll 1$)
In terms of $\eta = \nu\rho$, this is called the *Stokes approximation*:
$\eta\nabla^2\mathbf{v} = \nabla p$ (**Stokes approximation for** $\text{Re} \ll 1$)

TODO: Deriving the Stokes Formula

Consider stationary, incompressible flow past a sphere of radius R in a negligible gravitational field. Let u denote the relative velocity between sphere and steady flow.
Goal: determine drag force on sphere from viscous forces.
$\eta\nabla^2\mathbf{v} = \nabla p$ (Stokes approximation)
$F_{\text{drag}} = 6\pi\eta uR$ (result)

The Boundary Layer

Boundary condition for viscous fluids require vanishing flow velocity along boundary walls.
Conclusion: for viscous flow with large Reynolds numbers, which is approximately ideal far from boundaries, velocity vanishes rapidly in a thin region near boundary walls. This region of rapidly vanishing velocity is the *boundary layer*.

Laminar Boundary Layer

We will consider only laminar (not turbulent) boundary layers. Consider stationary, incompressible flow along a boundary plane at $y = 0$. Assume flow far from boundary is directionally uniform and moves in the $\hat{\mathbf{e}}_x$ direction.
Let $\mathbf{u} = u(x)\,\hat{\mathbf{e}}_x$ denote the velocity of the main flow
$\implies \nabla \times \mathbf{v} = \mathbf{0}$ in main flow (assuming uniform flow $\parallel \hat{\mathbf{e}}_x$)
$\implies$ main flow is potential/irrotational flow.
Assume translational invariance in the $\hat{\mathbf{e}}_z$ direction.
$\implies \mathbf{v} = v_x\,\hat{\mathbf{e}}_x + v_y\,\hat{\mathbf{e}}_y$ (assuming z invariance)

Simplifying the Navier-Stokes Equation

$(\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v}$ (the flow's NS equation)
$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 v_x}{\partial x^2} + \nu\frac{\partial^2 v_x}{\partial y^2}$ (in component form)
$v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v_y}{\partial x^2} + \nu\frac{\partial^2 v_y}{\partial y^2}$ (in component form)
$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$ (continuity equation for IF)
The flow in the boundary layer obeys...
$|v_x| \gg |v_y|$ (for steady flow in the $\hat{\mathbf{e}}_x$ direction)
$\frac{\partial v_i}{\partial y} \gg \frac{\partial v_i}{\partial x}$ (for steady flow in the $\hat{\mathbf{e}}_x$ direction)
$\frac{\partial p/\partial y}{\partial p/\partial x} \sim \frac{v_y}{v_x} \implies \frac{\partial p}{\partial x} \gg \frac{\partial p}{\partial y}$
$\frac{\partial p}{\partial x} \to \frac{\mathrm{d}p}{\mathrm{d}x}$ (letting $\frac{\partial p}{\partial y} \approx 0$ for $\frac{\partial p}{\partial x} \gg \frac{\partial p}{\partial y}$)
With the above approximations, the NS equation becomes...
$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} \approx -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \nu\frac{\partial^2 v_x}{\partial y^2}$ (x component)
$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} - \nu\frac{\partial^2 v_x}{\partial y^2} \approx -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}$ (x comp., rearranged)
$v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} = \nu\frac{\partial^2 v_y}{\partial y^2}$ (y component)

Prandtl Equations

In the irrotational, incompressible main flow, the flow obeys...
$\frac{p}{\rho} + \frac{u^2}{2} = \text{constant}$ (in main flow far from boundary layer)
$\implies -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = u\frac{\mathrm{d}u}{\mathrm{d}x}$ (after differentiating)
Substitute $-\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\mathrm{d}u}{\mathrm{d}x}$ into the x component of the above NS equation to get, together with the continuity equation...
$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} - \nu\frac{\partial^2 v_x}{\partial y^2} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = u\frac{\mathrm{d}u}{\mathrm{d}x}$ (**Prandtl equation**)
$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$ (**Prandtl equation**)

Boundary Conditions for the Prandtl Equations

(i) $\mathbf{v}(y)\big|_{y\to 0} = \mathbf{0}$ (velocity vanishes at boundary)

(ii) $v_x(y)\big|_{y\to\infty} = u$ (velocity approaches steady flow)

(iii) $v_y(y)\big|_{y\to\infty} = 0$ (velocity approaches steady flow)

Dimensionless Form of the Prandtl Equations

Let l and u_0 denote the problem's characteristic size and flow velocity, and introduce the dimensionless quantities...
$\text{Re} \equiv \frac{u_0 l}{\nu}$
$x' \equiv x/l$
$y' \equiv \sqrt{\text{Re}} \cdot y/l$
$v'_x = v_x/u_0$
$v'_y = \sqrt{\text{Re}} \cdot v_y/u_0$
$u' = u/u_0$
$\delta' = \sqrt{\text{Re}} \cdot \delta/l$ (dimensionless boundary layer width)
In terms of these quantities, the Prandtl equations become...
$v'_x\frac{\partial v'_x}{\partial x'} + v'_y\frac{\partial v'_x}{\partial y'} - \frac{\partial^2 v_x}{\partial y'^2} = u'\frac{\mathrm{d}u'}{\mathrm{d}x'}$
$\frac{\partial v'_x}{\partial x'} + \frac{\partial v'_y}{\partial y'} = 0$
Note: the equations are independent of ν, and so their solutions v'_x and v'_y are independent of Re!
$\implies v'_x$ and v'_y should be of order unity.
$v_x \sim u_0$ (assuming $v'_x \sim 1$)
$v_y \sim \frac{u_0}{\sqrt{\text{Re}}}$ (assuming $v'_y \sim 1$)
$\delta \sim \frac{l}{\sqrt{\text{Re}}}$ (assuming $\delta' \sim 1$)

Example: Boundary Layer Along a Semi-Infinite Plane

Consider stationary, incompressible flow along the semi-infinite plane given by $y = 0$ and $x > 0$. Let u be the constant velocity of uniform flow far from the plane.

$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$ (Prandtl eq. for constant u)

$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$ (Prandtl eq.)

$\mathrm{Re} = \frac{ul}{\nu}$

$x' = x/l$

$y' = \sqrt{\mathrm{Re}} \cdot y/l$

Complication: an infinite plane has no characteristic length l.
Resolution: v_x' and v_y' can depend only on a combination of x' and y' independent of l, for example...

$\xi \equiv y'/\sqrt{x'} = y\sqrt{\frac{u}{\nu x}}$ (dimensionless coordinate indpt. of l)

Estimating Boundary Layer Thickness

Plan: analyze behavior of v_x with increasing distance from plate.
$v_x \approx 0$ near the plate and $v_x/u \lesssim 1$ for increasing ξ or y.
Idea: estimate thickness δ of boundary layer as value of y where $v_x/u \sim 1$ and thus $\xi \sim 1$.

$\xi = y\sqrt{\frac{u}{\nu x}}$ (in general)

$1 \sim \delta\sqrt{\frac{u}{\nu x}}$ (setting $y \sim \delta$ at $\xi \sim 1$)

$\implies \delta \sim \sqrt{\frac{\nu x}{u}}$ (BL thickness scales as $\sqrt{x}$)

Blasius Equation

Flow is incompressible and two-dimensional, motivating the stream function formalism from Incompressible Flow in Two Dimensions.

$v_x = \frac{\partial \psi}{\partial y}$ and $v_y = -\frac{\partial \psi}{\partial x}$ (stream function)

$\psi = \sqrt{\nu x u} f(\xi)$ (ansatz for stream function)

$v_x(\xi) = \cdots = u f'(\xi)$ (using the chain rule)

$v_y(\xi) = \cdots = \frac{1}{2}\sqrt{\frac{\nu u}{x}}\left[\xi f'(\xi) - f(\xi)\right]$ (using the chain rule)

$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$ (Prandtl eq., for review)

Substitute $v_x(\xi)$ and $v_y(\xi)$ into the Prandtl eq. and get...

$2f''' + ff'' = 0$ (Blasius BL equation)

Boundary Conditions for the Blasius Equation

(i) $f'(\xi)\big|_{\xi=0} = 0$ (from $v_x(0) = 0$ and $v_x \propto f'(\xi)$)

(ii) $f(\xi)\big|_{\xi=0} = 0$ (from $v_y(0) = 0$, $v_z \propto f, f'$, and $f'(0) = 0$)

(iii) $f'(\xi)\big|_{\xi\to\infty} = 1$ (from $v_x(\infty) \to u$ and $v_x = uf'$)

TODO: Separation of Boundary Layer

Instabilities and Turbulence

Hydrodynamic instabilities and turbulence fall beyond the scope of this course; we will consider only a brief sampling of simpler topics.

Kelvin-Helmholtz Instabilities

Situation: two planar regions of ideal, incompressible fluid move relative to each other without mixing.
Align coordinate system so xy plane is boundary plane and $\hat{\mathbf{e}}_z$ is normal to boundary, pointing from fluid 2 into fluid 1.
Assume translational symmetry in the $\hat{\mathbf{e}}_y$ direction.

Let Φ_1 and Φ_2 denote velocity potentials in fluids 1 and 2.

$\nabla^2 \Phi_1 = 0$ (Laplace eq. for potential incompressible flow)

$\nabla^2 \Phi_2 = 0$ (Laplace eq. for potential incompressible flow)

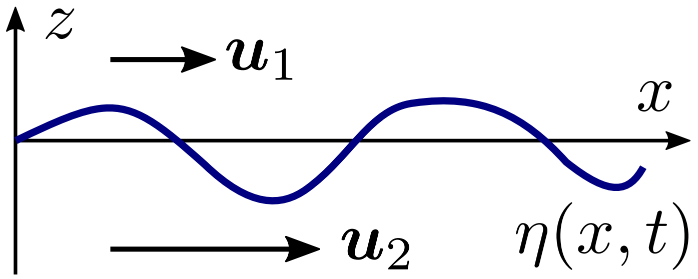

Figure 5: Geometry of Kelvin-Helmholtz instabilities.

Let $\eta(x,t)$ denote the boundary profile between fluids.
Let $\mathbf{u}_1$ and $\mathbf{u}_2$ denote main flow velocities far from boundary.
Let $\mathbf{v}_1$ and $\mathbf{v}_2$ denote flow velocities along boundary profile.

$\frac{\partial \eta}{\partial t} = -\frac{\partial \eta}{\partial x} v_{1_x} + v_{1_z} = -\frac{\partial \eta}{\partial x} v_{2_x} + v_{2_z}$

Justification: profile η decreases where $v_{1_x} > 0$ and $\frac{\partial \eta}{\partial x} > 0$.

Euler Equations of Motion

Flow is assumed ideal, so we use the Euler equation.

$\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 = \mathbf{g} - \frac{1}{\rho_1}\nabla p_1$ (the flow's Euler eq.)

Align coordinate system so that $\mathbf{g} = -g\,\hat{\mathbf{e}}_z$.

$\implies \mathbf{g} = -g\nabla z = -\nabla(gz) = -\nabla(g\eta)$ (constant g and $z = \eta$)

$(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla^2 v^2 - \mathbf{v} \times (\nabla \times \mathbf{v})$ (general vector identity)

$= \frac{1}{2}\nabla^2 v^2$ (for potential flow $\nabla \times \mathbf{v} = \mathbf{0}$)

$\mathbf{v}_1 = \nabla \Phi_1$ (for potential flow)

Apply $\mathbf{g} = -\nabla(g\eta)$, $\mathbf{v} = \nabla\Phi$, and vector identity to get...

$\frac{\partial(\nabla\Phi_1)}{\partial t} + \frac{1}{2}\nabla v_1^2 = -\nabla\left(g\eta + \frac{p_1}{\rho_1}\right)$ (Euler eq. in terms of ∇)

$\nabla\left(\frac{\partial \Phi_1}{\partial t} + \frac{v_1^2}{2} + g\eta + \frac{p_1}{\rho_1}\right) = 0$ (after rearranging)

$\implies \frac{\partial \Phi_1}{\partial t} + \frac{v_1^2}{2} + g\eta + \frac{p_1}{\rho_1} = \text{constant} \equiv \frac{p_0}{\rho_1} + \frac{u_1^2}{2}$

$\frac{\partial \Phi_2}{\partial t} + \frac{v_2^2}{2} + g\eta + \frac{p_2}{\rho_2} = \frac{p_0}{\rho_2} + \frac{u_2^2}{2}$ (analogously in region 2)

p_0 is steady flow reference pressure far from boundary profile.

Linearizing the Equations of Motion

Decompose Φ_1 and Φ_2 into steady-flow and perturbative terms:

$\Phi_1(x,z,t) = u_1 x + \phi_1(x,z,t)$

$\Phi_2(x,z,t) = u_2 x + \phi_2(x,z,t)$

$\implies \mathbf{v}_1 = \nabla\Phi_1 = u_1\,\hat{\mathbf{e}}_x + \nabla\phi_1$

$\implies \mathbf{v}_2 = \nabla\Phi_2 = u_2\,\hat{\mathbf{e}}_x + \nabla\phi_2$

Linearizing Boundary Profile Equation

$\frac{\partial \eta}{\partial t} = -\frac{\partial \eta}{\partial x} v_{1_x} + v_{1_z} = -\frac{\partial \eta}{\partial x} v_{2_x} + v_{2_z}$ (profile height for review)

Substitute in $\mathbf{v}_1$ and $\mathbf{v}_2$ into above equations and get...

$\frac{\partial \eta}{\partial t} = -\frac{\partial \eta}{\partial x}\left(u_1 + \frac{\partial \phi_1}{\partial x}\right) + \frac{\partial \phi_1}{\partial z} = -\frac{\partial \eta}{\partial x}\left(u_2 + \frac{\partial \phi_2}{\partial x}\right) + \frac{\partial \phi_2}{\partial z}$

Neglect the second-order $\mathcal{O}(\eta\phi)$ perturbation terms to get...

$\frac{\partial \eta}{\partial t} \approx -\frac{\partial \eta}{\partial x} u_1 + \frac{\partial \phi_1}{\partial z} \approx -\frac{\partial \eta}{\partial x} u_2 + \frac{\partial \phi_2}{\partial z}$ (neglecting $\mathcal{O}(\eta\phi)$)

Linearizing the Euler Equation

$\frac{\partial \Phi_1}{\partial t} + \frac{v_1^2}{2} + g\eta + \frac{p_1}{\rho_1} = \frac{p_0}{\rho_1} + \frac{u_1^2}{2}$ (Euler eq. in region 1)

$\frac{\partial \Phi_1}{\partial t} = \frac{\partial}{\partial t}\left[u_1 x + \phi_1(x,z,t)\right] = \frac{\partial \phi_1}{\partial t}$ (exact)

$v_1^2 = \left[\left(u_1 + \frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial z}\right)^2\right]$ (exact)

$\approx u_1^2 + 2u_1\frac{\partial \phi_1}{\partial x}$ (neglecting $\mathcal{O}(\phi^2)$)

$\frac{\partial \phi_1}{\partial t} + \frac{u_1^2}{2} + u_1\frac{\partial \phi_1}{\partial x} + g\eta + \frac{p_1}{\rho_1} = \frac{p_0}{\rho_1} + \frac{u_1^2}{2}$ (linearized)

$\rho_1\left(\frac{\partial \phi_1}{\partial t} + u_1\frac{\partial \phi_1}{\partial x} + g\eta\right) = p_0 - p_1$ (after simplifying)

$\rho_2\left(\frac{\partial \phi_2}{\partial t} + u_2\frac{\partial \phi_2}{\partial x} + g\eta\right) = p_0 - p_2$ (analogously in region 2)

Subtract equations at boundary profile where $p_1 = p_2$ to get...

$\rho_1\left(\frac{\partial \phi_1}{\partial t} + u_1\frac{\partial \phi_1}{\partial x} + g\eta\right) = \rho_2\left(\frac{\partial \phi_2}{\partial t} + u_2\frac{\partial \phi_2}{\partial x} + g\eta\right)$

Dispersion Relation

In one place the equations connecting ϕ_1, ϕ_2 and η are...

$\frac{\partial \eta}{\partial t} = -\frac{\partial \eta}{\partial x} u_1 + \frac{\partial \phi_1}{\partial z}$

$\frac{\partial \eta}{\partial t} = -\frac{\partial \eta}{\partial x} u_2 + \frac{\partial \phi_2}{\partial z}$
$\rho_1 \left(\frac{\partial \phi_1}{\partial t} + u_1 \frac{\partial \phi_1}{\partial x} + g\eta \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} + u_2 \frac{\partial \phi_2}{\partial x} + g\eta \right)$
Idea: the equations are linear in ϕ_1, ϕ_2 and η, so we search for wave solutions of given wavelength and write entire solution as a superposition of monochromatic waves.
$\phi_1 = A_1 e^{i(kx-\omega t)} e^{-kz}$ (ansatz)
$\phi_2 = A_2 e^{i(kx-\omega t)} e^{+kz}$ (ansatz)
$\eta = \eta_0 e^{i(kx-\omega t)}$ (ansatz)
Substitute ansatzes into above three eqs. at $z=0$ and get...
$-i\omega\eta_0 = -ik\eta_0 u_1 - kA_1$ (from first equation)
$-i\omega\eta_0 = -ik\eta_0 u_2 + kA_2$ (from second equation)
$\rho_1(-i\omega A_1 + ikA_1 u_1 + g\eta_0) = \rho_2(-i\omega A_2 + ikA_2 u_2 + g\eta_0)$ (3rd)

$$\begin{pmatrix} k & 0 & i(ku_1-\omega) \\ 0 & -k & i(ku_2-\omega) \\ i\rho_1(ku_1-\omega) & -i\rho_2(ku_2-\omega) & g(\rho_1-\rho_2) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \eta_0 \end{pmatrix} = \mathbf{0}$$

For a nontrivial solution, require $\det \mathbf{M} = 0$...
$0 \equiv \det \mathbf{M} = k^2 g(\rho_1-\rho_2) - (ku_2-\omega)^2\rho_1 k + k\rho_2(ku_2-\omega)^2$
$\quad = (\rho_1+\rho_2)\frac{\omega^2}{k^2} - 2(\rho_1 u_1 \rho_2 u_2)\frac{\omega}{k} + \left[\rho_1 u_1^2 + \rho_2 u_2^2 + \frac{g}{k}(\rho_1-\rho_2)\right]$
$\frac{\omega}{k} = \frac{\rho_1 u_1 + \rho_2 u_2}{\rho_1+\rho_2} \pm \sqrt{\frac{(\rho_1 u_1+\rho_2 u_2)^2}{(\rho_1+\rho_2)^2} - \frac{\rho_1 u_1^2 + \rho_2 u_2^2 + \frac{g}{k}(\rho_1-\rho_2)}{\rho_1+\rho_2}}$ (dispersion relation)

Stability Criterion

Idea: complex frequency components in above dispersion relation will cause ansatzes for ϕ_1, ϕ_2 and η to grow exponentially.
Conclusion: for stability, ω/k must be real, so square root's argument must be positive. For *instability...*
$\implies \frac{(\rho_1 u_1+\rho_2 u_2)^2}{(\rho_1+\rho_2)^2} < \frac{\rho_1 u_1^2 + \rho_2 u_2^2 + \frac{g}{k}(\rho_1-\rho_2)}{\rho_1+\rho_2}$ (instability criterion)
$(u_1-u_2)^2 > \frac{g}{k}\frac{\rho_2^2-\rho_1^2}{\rho_1\rho_2}$ (after rearranging)
Conclusion: there always exists a large enough k for instability.

Accounting for Surface Tension

A more general analysis, accounting for surface tension, letting γ denote fluid's surface tension coefficient, gives...
$(u_1-u_2)^2 > \frac{g}{k}\frac{\rho_2^2-\rho_1^2}{\rho_1\rho_2} + \gamma k \frac{\rho_1+\rho_2}{\rho_1\rho_2}$ (for instability)
$k_{\min} = \sqrt{\frac{g(\rho_2-\rho_1)}{\gamma}}$ (minimum of RHS)
$(u_1-u_2)^2\big|_{k=k_{\min}} = 2\sqrt{g\gamma}\frac{\rho_2^2-\rho_1^2}{\rho_1\rho_2\sqrt{\rho_2-\rho_1}}$ (minimum speed difference below which instabilities don't occur)

Reynolds Stress Tensor

Idea: model turbulent flow by decomposing velocity and pressure into steady and fluctuating components.
$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}'$ (decomposition into steady and fluctuating)
$p = \overline{p} + p'$ (decomposition into steady and fluctuating)
$\overline{\mathbf{v}'} = \mathbf{0}$ and $\overline{p'} = 0$ (unsteady comps. average to zero)
Restriction: assume flow is incompressible and stationary.
$\implies \nabla \cdot \mathbf{v} = \nabla \cdot (\overline{\mathbf{v}} + \mathbf{v}') = 0$ (continuity equation for IF)
$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v}$ (NS equation for steady, IF)
$v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\eta}{\rho}\frac{\partial^2}{\partial x_j^2} v_i$ (in component form)
$(\overline{v}_j + v_j')\frac{\partial(\overline{v}_i + v_i')}{\partial x_j} = -\frac{1}{\rho}\frac{\partial(\overline{p}+p')}{\partial x_i} + \frac{\eta}{\rho}\frac{\partial^2}{\partial x_j^2}(\overline{v}_i + v_i')$ (decomposed)
$\overline{v}_j \frac{\partial \overline{v}_i}{\partial x_j} + \overline{v_j' \frac{\partial v_i'}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\eta}{\rho}\frac{\partial^2}{\partial x_j^2}\overline{v}_i$ (averaged over time)
(Quadratic fluctuating terms may have a non-zero average)
$\frac{\partial}{\partial x_j}\left(\overline{v_i' v_j'}\right) = \overline{v_i' \frac{\partial v_j'}{\partial x_j}} + \overline{v_j' \frac{\partial v_i'}{\partial x_j}}$ (deriving identity for later use)
$\quad = \overline{v_j' \frac{\partial v_i'}{\partial x_j}}$ (from continuity equation $\frac{\partial v_j}{\partial x_j} = 0$)
$\overline{v}_j \frac{\partial \overline{v}_i}{\partial x_j} + \overline{v_j' \frac{\partial v_i'}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\eta}{\rho}\frac{\partial^2}{\partial x_j^2}\overline{v}_i$ (review: averaged NS eq.)
$\rho\overline{v}_j \frac{\partial \overline{v}_i}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \eta\frac{\partial^2}{\partial x_j^2}\overline{v}_i - \rho\overline{v_j' \frac{\partial v_i'}{\partial x_j}}$ (rearranged)
$\rho\overline{v}_j \frac{\partial \overline{v}_i}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\eta\frac{\partial \overline{v}_i}{\partial x_j} - \rho\overline{v_i' v_j'}\right)$ (using above identity)
$\rho\overline{v}_j \frac{\partial \overline{v}_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-\overline{p}\delta_{ij} + \eta\frac{\partial \overline{v}_i}{\partial x_j} - \rho\overline{v_i' v_j'}\right)$ (using $\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial x_j}\delta_{ij}$)
The RHS motivates the introduction of a "total" stress tensor:

$$p_{ij}^{\text{tot}} = -\underbrace{\overline{p}\delta_{ij}}_{\text{isotropic}} + \underbrace{\eta\left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i}\right)}_{\text{viscous}} - \underbrace{p\overline{v_i' v_j'}}_{\text{Reynolds}}$$

The term $p\overline{v_i' v_j'}$ is called the *Reynolds stress tensor* and represents the time-averaged correlations of turbulent fluctuations.