Solved Exercises in Radiation Measurement

Notes from the Exercises component of the third-year undergraduate course *Merjenje ionizirajočega sevanja* (Measurement of Ionizing Radiation), led by dr. Luka Šantelj at the Faculty of Mathematics and Physics at the University of Ljubljana in the academic year 2020-21. Credit for the material covered in these notes is due to dr. Šantelj, while the voice, typesetting, and translation to English in this document are my own.

Disclaimer: This document will inevitably contain some mistakes—both simple typos and legitimate errors. Keep in mind that these are the notes of an undergraduate student in the process of learning the material himself, so take what you read with a grain of salt. If you find mistakes and feel like telling me, I will be grateful and happy to hear from you, even for the most trivial of errors. You can reach me by email, in English, Slovene, or Spanish, at ejmastnak@gmail.com.

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Contents

1 First Exercise Set **4**
- 1.1 Review: Differential Scattering Cross Section . . . 4
- 1.2 A Theoretical Dark Matter Scattering Experiment . . . 5
- 1.3 Derivation: Energy Lost in a Heavy Particle-Electron Collision . . . 6

2 Second Exercise Set **8**
- 2.1 Derivation: A Heavy Particle's Ionizing Energy Losses in Matter . . . 8
- 2.2 Numerical Values of Ionizing Losses in Argon . . . 12

3 Third Exercise Set **13**
- 3.1 Proton Therapy (Ionizing Energy Losses in Water) . . . 13
- 3.2 Theory: Photon Interaction with Matter . . . 16
- 3.3 Compton Scattering Attenuation in Sodium Iodide . . . 17

4 Fourth Exercise Set **19**
- 4.1 Compton Scattering . . . 19
- 4.2 Theory: Compton Scattering Formulae . . . 20
- 4.3 Compton Scattering in Sodium Iodide . . . 21
- 4.4 Theory: Signal in a Rectangular Ionization Cell . . . 22
- 4.5 Voltage Drop in a Rectangular Ioniziation Cell . . . 23

5 Fifth Exercise Set **25**
- 5.1 Theory: Multiplication Factor in a Parallel-Plate Cell . . . 25
- 5.2 Voltage for a Given Multiplication Factor in a Parallel-Plate Cell . . . 25
- 5.3 Theory: Cylindrical Ionization Chamber . . . 26
- 5.4 Voltage for a Given Multiplication Factor in a Cylindrical Chamber . . . 27
- 5.5 Time Delay Between Ionization Events in a Cylindrical Chamber . . . 28

6 Sixth Exercise Set **30**
- 6.1 Signal Dynamics in a Cylindrical Ionization Chamber . . . 30
- 6.2 Energy Resolution in Rectangular Ionization Cell . . . 35

7 Seventh Exercise Set **37**
- 7.1 Theory: Measuring Momentum . . . 37
- 7.2 Identifying Mass from Momentum and Energy Measurements . . . 38

8 Eighth Exercise Set **40**
- 8.1 Theory: Important Quantities in Intrinsic Semiconductors . . . 40
- 8.2 Resistivity of an Intrinsic and Doped Semiconductor . . . 41
  - 8.2.1 Resistivity of an Intrinsic Semiconductor . . . 42
  - 8.2.2 Resistivity of a Doped Semiconductor . . . 42
- 8.3 Depletion Region Width in a p-n Junction . . . 43

9 Ninth Exercise Set **45**
- 9.1 Theory: Relating Resistivity and Depletion Region Width . . . 45
- 9.2 Energy Resolution in a Semiconducting Detector . . . 46
- 9.3 Signal Dynamics in a p-n Semiconducting Detector . . . 47

9.3.1 Electrical Current 47
9.3.2 Voltage Signal 49
9.3.3 Limit Cases of Electron Signal 52

10 Tenth Exercise Set **53**
10.1 Theory: Position-Sensitive Semiconducting Detectors 53
10.2 Theory: Statistics 54
10.2.1 Deriving the Poisson Distribution 54
10.2.2 Statistical Significance 55

11 Eleventh Exercise Set **56**
11.1 Statistical Significance in a Radioactive Decay Experiment 56
11.2 Lower Bound on Decay Time at a 90% Confidence Level 57
11.3 Signal Dynamics in a Scintillating Detector 58

12 Twelfth Exercise Set **61**
12.1 Efficiency of a CsI(Tl) Scintillator 61
12.2 Theory: Effect of Fluctuations on Photomultiplier Amplification . . . 61
12.3 Post-Scattering Energy Distribution of Fast Neutrons 64

13 Thirteenth Exercise Set **67**
13.1 Slowing Neutrons to Thermal Energy 67
13.2 Probability for Interaction in a ^{3}He Neutron Detector 68
13.3 Theory: Cherenkov Radiation 70
13.3.1 Threshold Speed and Momentum 70
13.3.2 Cherenkov Photon Frequency Spectrum 70
13.3.3 Cherenkov Photon Wavelength Spectrum 71
13.4 Detected Photons in a Cherenkov Detector 71

14 Fourteenth Exercise Set **73**
14.1 A Threshold Cherenkov Radiation Detector 73
14.1.1 Finding Momentum for a Given Pion Detection Efficiency . . 73
14.1.2 Classifying Kaons and Pions 75
14.2 Ring-Imaging Cherenkov Detector 77

1 First Exercise Set

1.1 Review: Differential Scattering Cross Section

We begin by reviewing the differential scattering cross section, which will be useful in the future for describing the interaction of ionizing particles with matter. The differential cross section is in general a function of both energy and solid angle, and is defined as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{F}\frac{\mathrm{d}N_{\mathrm{s}}}{\mathrm{d}\Omega},$$

where F is the flux of particles and N_{s} is the number of particles scattered into a solid angle $\mathrm{d}\Omega$ per unit time and has units time^{-1}.

The unit of flux is $[\mathrm{time}^{-1}\,\mathrm{length}^{-2}]$ (per time per area). If follows that $\mathrm{d}\sigma$ has units of area, and can be interpretted as the size of a target. Usually the scattering cross sections are given in barns, where $1\,\mathrm{b} = 10^{-28}\,\mathrm{m}^{-2}$.

Sketch: beam of particles (1) incident on a second particle (2). As long as there is an interaction between the particles (1) and (2), then particles (1) scatter from (2).

We find total scattering cross section by integrating the differential cross section over all solid angle

$$\sigma_{\mathrm{tot}}(E) = \iint_{\Omega} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega.$$

Typical configuration: a flux of particles incident on a target of width δx and cross-sectional area S. The number of scattered particles N_{s} in some solid angle Ω per unit time is

$$N_{\mathrm{s}}(\Omega) = F \cdot S \cdot n_{\mathrm{t}} \cdot \delta x \cdot \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega},$$

where n_{t} is the density of scattering centers within the target. For example, for nuclear scattering, n_{t} might be the number of atomic nuclei per unit volume in the target. The total number of scattered particles in the entire solid angle per unit time is

$$N_{\mathrm{s}} = F S n_{\mathrm{t}} \delta x \sigma_{\mathrm{tot}}.$$

Lesson: if you know the cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ corresponding to a single incident particle scattering from a single, isolated nucleus, you can generalize the result to find the number of particles scattered from a macroscopic scatterer containing many nuclei.

Next, we note that $n_{\mathrm{t}}\delta x\sigma_{\mathrm{tot}}$ is just the probability that a single incident particle scatters from the target in the region δx.

The above expressions for the number of scattered particles N_{s} holds only for small δx. For a thick target, the beam intensity attenuates exponentially with the distance traveled through the target.

To derive this, we ask what is the probability that a particle does *not* scatter up to a certain thickness (distance traveled through the target), e.g. $x + \mathrm{d}x$. We denote this probability by

$$P(x + \mathrm{d}x) = P(x) \cdot \text{probabiity for not scattering} = P(x)(1 - n_{\mathrm{t}}\sigma_{\mathrm{tot}}\,\mathrm{d}x).$$

We then expand the left hand side (which is accurate for intinitesimal $\mathrm{d}x$) and distribute the right hand side

$$P(x) + \frac{\mathrm{d}P}{\mathrm{d}x}\,\mathrm{d}x = P(x) - P(x) n_\mathrm{t} \sigma_\mathrm{tot}\,\mathrm{d}x.$$

The result is

$$\frac{\mathrm{d}P}{\mathrm{d}x} = -P(x) n_\mathrm{t} \sigma_\mathrm{tot} \implies P(x) = C e^{-n_\mathrm{t}\sigma_\mathrm{tot} x}.$$

From the condition $P(0) = 1$ we have $C = 1$. This is the probability that a particle survives up to the distance x traveled through the target. Meanwhile, the probability for scattering before x is simply $1 - P(x)$.

Next we introduce the concept of mean free path: average path traveled by a particle in the target before scattering. We will denote it λ. If we know $P(x)$, we can then calculate the mean free path using the general definition

$$\lambda = \frac{\int x P(x)\,\mathrm{d}x}{\int P(x)\,\mathrm{d}x} = \frac{1}{n_\mathrm{t}\sigma_\mathrm{tot}}.$$

As expected, the larger the density of scatterers and cross sections, the smaller the mean free path (the less far the particle travels through the material). Example: if you know σ_tot for a photon in water, you can find how far you can see in water (how far light travels.)

1.2 A Theoretical Dark Matter Scattering Experiment

Background: In certain hypothesized theories, dark matter is formed of weakly interacting massive particles (WIMPS). We aim to detect WIMPS from their scattering interactions with heavy xenon particles, which emit detectable photons. Assume the universe is permeated with dark matter WIPMS at a number density $n_\mathrm{wimp} = 10^4\,\mathrm{m}^{-3}$. *As an estimate of detector speed through hypothesized dark matter, the speed of our solar system through the Milky Way galaxy is* $v = 2.2 \cdot 10^5\,\mathrm{m\,s}^{-1}$.

Question: With what mass of xenon m_Xe *should we fill a detector such that we detect* $N_\mathrm{s} = 10$ *WIMPS after 5 years of measurement? The estimated cross section for WIMP scattering on atomic nuclei is* $\sigma_\mathrm{tot} = 10^{-42}\,\mathrm{cm}^2$*, the molar mass of xenon is* $M = 131.3\,\mathrm{g\,cm}^{-3}$.

We find the total number of particles scattered in the time t with

$$N_\mathrm{s} = F S n_\mathrm{Xe} \sigma \cdot \delta x \cdot t, \tag{1.1}$$

where n_Xe is the density of atomic nuclei in the xenon sample and S is the detector's cross-sectional area with respect to the direction of the detector's motion through the dark matter.

Our first step is to find expressions for each of the individual terms in Equation 1.1 in terms of the given data.

We estimate the flux F of incident WIMPS on the detector with

$$F = v n_\mathrm{wimp} = \left(2.2 \cdot 10^5\,\mathrm{m\,s}^{-1}\right) \cdot \left(10^4\,\mathrm{m}^{-3}\right) = 2.2 \cdot 10^9\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}.$$

We now find volume density of atomic nuclei in the xenon detector with

$$n_{\mathrm{Xe}} = \frac{\rho_{\mathrm{Xe}} N_{\mathrm{A}}}{M_{\mathrm{Xe}}},$$

where N_A is Avogadro's number.

Finally, we note that the expression $S \cdot \delta x$ in Equation 1.1 is just the detector's volume, which we can write in terms of the constituent xenon's mass and density as

$$S \cdot \delta x = V = \frac{m_{\mathrm{Xe}}}{\rho_{\mathrm{Xe}}}.$$

Putting the pieces together, Equation 1.1 for the number of scattered WIMPS becomes

$$N_{\mathrm{s}} = FSn_{\mathrm{Xe}}\sigma \cdot \delta x \cdot t = (vn_{\mathrm{wimp}}) \cdot \frac{m_{\mathrm{Xe}}}{\rho_{\mathrm{Xe}}} \cdot \frac{\rho_{\mathrm{Xe}} N_A}{M_{\mathrm{Xe}}} \sigma \cdot t = (vn_{\mathrm{wimp}}) \frac{N_A \sigma \cdot t}{M_{\mathrm{Xe}}} \cdot m_{\mathrm{Xe}}.$$

We then rearrange to find the required xenon mass:

$$\begin{aligned} m_{\mathrm{Xe}} &= \frac{N_S M_{\mathrm{Xe}}}{N_A (vn_{\mathrm{wimp}}) \cdot \sigma \cdot t} \\ &= \frac{10 \cdot (131\,\mathrm{g\,mol^{-1}})}{(6 \cdot 10^{23}\,\mathrm{mol^{-1}}) \cdot (2.2 \cdot 10^{9}\,\mathrm{m^{-2}\,s^{-1}}) \cdot (10^{-46}\,\mathrm{m^2}) \cdot (15 \cdot 10^{7}\,\mathrm{s})} \\ &\approx 6 \cdot 10^3\,\mathrm{kg}. \end{aligned}$$

1.3 Derivation: Energy Lost in a Heavy Particle-Electron Collision

Derive an expression for the maximum energy lost by an ionizing particle of mass m *and momentum* $\mathbf{p}$ *when scattering from a stationary electron with mass* m_{e}*. Assume the ionizing particle is heavy, i.e.* $m \gg m_{\mathrm{e}}$*.*

Let $\mathbf{p}'$ and $\mathbf{p}_{\mathrm{e}}$ denote the heavy particle and electron momenta after scattering, respectively. We begin with the equation for conservation of total relativistic energy before and after scattering, which reads

$$E_{\mathrm{before}} = E_{\mathrm{after}} \implies \sqrt{p^2c^2 + m^2c^4} + m_{\mathrm{e}}c^2 = \sqrt{(p')^2c^2 + m^2c^4} + T_{\mathrm{e}} + m_{\mathrm{e}}c^2, \quad (1.2)$$

where T_{e} is the electron's kinetic energy after the collision and is related to the electron's mass and momentum by the canonical relativstic mass-energy relationship

$$T_{\mathrm{e}} + m_{\mathrm{e}}c^2 = \sqrt{p_{\mathrm{e}}^2c^2 + m_{\mathrm{e}}^2c^4}. \quad (1.3)$$

Meanwhile, the equation for conservation of momentum reads

$$\mathbf{p} = \mathbf{p}' + \mathbf{p}_{\mathrm{e}} \implies \mathbf{p}' = \mathbf{p} - \mathbf{p}_{\mathrm{e}}.$$

We then square this relationship to produce

$$(p')^2 = p^2 + p_{\mathrm{e}}^2 - 2pp_{\mathrm{e}}\cos\theta \quad (1.4)$$

where θ is the angle between the heavy ionizing particle's initial momentum $\mathbf{p}$ and the scattered electron momentum $\mathbf{p}_{\mathrm{e}}$.

We begin by solving Equation 1.3 for the electron momentum p_e, which leads to

$$T_e + m_e c^2 = \sqrt{p_e^2 c^2 + m_e^2 c^4} \implies p_e^2 = \frac{(T_e + m_e c^2)^2 - m_e^2 c^4}{c^2}.$$

We then substitute this expression for p_e into Equation 1.4 and multiply through by c^2 to get

$$(p')^2 c^2 = p^2 c^2 + \left(T_e + m_e c^2\right)^2 - m_e^2 c^4 - 2pc\cos\theta\sqrt{(T_e + m_e c^2)^2 - m_e^2 c^4}, \qquad (1.5)$$

which relates the ionizing particle's final momentum p' to the electron energy T_e.

Next, we solve the energy conservation equation (Eq. 1.2), which reads

$$\sqrt{p^2c^2 + m^2c^4} = \sqrt{(p')^2c^2 + m^2c^4} + T_e,$$

for the final particle momentum p', which produces

$$(p')^2 c^2 = p^2c^2 + T_e^2 - 2T_e\sqrt{p^2c^2 + m^2c^4}.$$

We then substitute in Equation 1.5 for $p'c$ and combine like terms to get

$$(T_e + m_e c^2)^2 - m_e c^4 - 2pc\cos\theta\sqrt{(T_e + m_e c^2)^2 - m_e^2 c^4} = T_e^2 - 2T_e\sqrt{p^2c^2 + m^2c^4}.$$

All that remains is to solve for T_e. After some tedious but straightforward algebra, the result comes out to

$$T_e = \frac{2m_e c^2 p^2 c^2 \cos^2\theta}{\left(m_e c^2 + \sqrt{p^2c^2 + m^2c^4}\right)^2 - p^2c^2\cos^2\theta}.$$

Maximum energy loss occurs in a head-on collision, in which case we have $\theta = 0$, $\cos\theta = 1$ and kinetic energy

$$T_{\max} = \frac{2m_e p^2}{\left(m_e + \sqrt{p^2 + m^2}\right)^2 - p^2}.$$

Finally, to see the problem from a different perspective, we write the ionizing particle's momentum p in terms of its gamma factor, i.e.

$$p = \gamma m v = \gamma m \beta c.$$

In terms of the γ and β factors, the maximum energy reads

$$T_{\max} = \frac{2m_e c^2 \gamma^2 \beta^2}{1 + 2\frac{m_e}{m}\gamma + \left(\frac{m_e}{m}\right)^2}.$$

If the ionizing particle is much more massive than the electron, the maximum energy loss reduces to

$$T_{\max} \to 2m_e c^2 \gamma^2 \beta^2 \qquad (1.6)$$

As an example, we consider a proton with momentum $pc = 2\,\text{GeV}$, which gives

$$\gamma\beta = \frac{pc}{mc^2} \approx \frac{2\,\text{GeV}}{1\,\text{GeV}} = 2.$$

The corresponding maximum energy loss in a head-on collision is

$$T_{\max} = 2 \cdot m_e c^2 \cdot 4 = 8m_e c^2 \sim 4\,\text{MeV},$$

which is relatively small compared to the proton's roughly 2 GeV energy.

2 Second Exercise Set

2.1 Derivation: A Heavy Particle's Ionizing Energy Losses in Matter

Derive the average energy loss $-\frac{\mathrm{d}E}{\mathrm{d}x}$ *(formally* $-\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle$*) of a heavy ionizing particle (mass* $m \gg m_e$*) per unit distance traveled through matter.*

We begin by parameterizing the particle's energy loss in terms of the average kinetic energy $\overline{T}$ gained by an electron in a single collision with the ionizing particle; the expression reads

$$-\,\mathrm{d}E = (n_\mathrm{a} Z_\mathrm{m} \sigma \,\mathrm{d}x) \cdot \overline{T},$$

where n_a is the number density of atoms in the material through which the ionizing particle travels, Z_m is the material's atomic number, and σ is the cross section for interaction of the ionizing particle with electrons in the material.

Interpretation: the expression $n_\mathrm{a} Z_\mathrm{m} \sigma \,\mathrm{d}x$ is just the total number of interactions between the ionizing particle and electrons in a distance $\mathrm{d}x$; so the total energy loss by the particle is just the negative of the number of interactions times the average gained $\overline{T}$ by an electron per interaction.

Our goal is to find $\overline{T}$; we'll do this using the scattering cross section. We first write

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = n_\mathrm{a} Z_\mathrm{m} \int_{T_\mathrm{min}}^{T_\mathrm{max}} T \frac{\mathrm{d}\sigma}{\mathrm{d}T} \,\mathrm{d}T, \tag{2.1}$$

where the integral is the formal definition of the average energy $\overline{T}$, and runs from the minimum to the maximum possible energy gained by an electron per collision.

The above expression essentially reduces our problem to finding the cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}T}$. We have a neat trick to do this: The scattering of a heavy particle from an electron is governed by the same dynamics as an electron scattering from a heavy nucleus, just in reverse. In other words, we can reuse the results for the scattering of electrons from a heavy nucleus (i.e. the well-known Rutherford cross section), and reverse the result to apply to a heavy particle scattering from an electron.

Finding the Ionization Cross Section

To reiterate, our goal is to find the cross section encoding the interaction of the ionizing particle with electrons in the matter; we will do this by reversing the Rutherford scattering cross section. Without derivation, the Rutherford cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{Z_\mathrm{p} e_0^2}{8\pi\epsilon_0 p v}\right)^2 \frac{1}{\sin^4(\theta/2)}.$$

To be clear, this cross section describes an electron with momentum p and velocity v scattering from a massive particle with atomic number Z_p at an angle θ relative to the direction of incidence.

We begin by writing the electron's momentum four vectors before and after the collision as p^μ and p'^μ respectively, and write the transfer of momentum during the collision as

$$q^\mu \equiv p'^\mu - p^\mu.$$

Our plan is to work in terms of the Lorentz-invariant quantity

$$(q^\mu)^2 = (p'^\mu - p^\mu)^2,$$

which, being invariant across all Lorentz frames, will allow use to transform the Rutherford cross section for an electron scattering from a heavy particle into the cross section for a heavy particle scattering from an electron.

If we use θ to denote the angle between the initial and final electron momentum $\mathbf{p}$ and $\mathbf{p}'$, the transfer of momentum may be written

$$\frac{q}{2} = p \sin\frac{\theta}{2},$$

which we've written in terms of a half angle to match the exprssion $\sin^4\theta/2$ in the Rutherford cross section. Rearranging and squaring both sides, we get

$$\sin^2\frac{\theta}{2} = \left(\frac{q}{2p}\right)^2 \implies \frac{1}{\sin^4\frac{\theta}{2}} = 16\frac{p^4}{q^4}. \tag{2.2}$$

In terms of q, the Rutherford cross section reads

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{Z_\mathrm{p} e_0^2}{8\pi\epsilon_0 p v}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}} = \left(\frac{Z_\mathrm{p} e_0^2}{8\pi\epsilon_0 p v}\right)^2 \frac{16p^4}{q^4}.$$

TODO: is the momentum in the Rutherford cross section meant as a four vector? Where do 3 vectors end and four vectors start?

Next, we aim to make a change of variables from $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ to $\frac{\mathrm{d}\sigma}{\mathrm{d}q^2}$. We do this with the chain rule:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}q^2} = \frac{\mathrm{d}\sigma}{\mathrm{d}q^2},$$

which means we need an expression for $\frac{\mathrm{d}\Omega}{\mathrm{d}q^2}$. We first write $\mathrm{d}\Omega$ as a differential of solid angle, i.e. $\mathrm{d}\Omega = 2\pi \sin\theta \,\mathrm{d}\theta$. From Equation 2.2, the quantity q^2 comes out to

$$q^2 = 4p^2 \sin^2\frac{\theta}{2},$$

while the derivative of q with respect to θ is

$$\frac{\mathrm{d}q^2}{\mathrm{d}\theta} = 4p^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2p^2\sin\theta \implies \sin\theta\,\mathrm{d}\theta = \frac{\mathrm{d}q^2}{2p^2}.$$

We then combine this with the general identity $\mathrm{d}\Omega = 2\pi\sin\theta\,\mathrm{d}\theta$ to get

$$\mathrm{d}\Omega = 2\pi\sin\theta\,\mathrm{d}\theta = 2\pi\cdot\left(\frac{\mathrm{d}q^2}{2p^2}\right) = \pi\frac{\mathrm{d}q^2}{p^2} \implies \frac{\mathrm{d}\Omega}{\mathrm{d}q^2} = \frac{\pi}{p^2}.$$

TODO: how are we sure the angle θ between momenta is the same angle as the differential of solid angle?

Using the just-derived expression for $\frac{\mathrm{d}\Omega}{\mathrm{d}q^2}$ with the Rutherford cross section and the expression for $\csc^4\frac{\theta}{2}$ in Equation 2.2, the desired expression for $\frac{\mathrm{d}\sigma}{\mathrm{d}q^2}$ is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}q^2} = \left(\frac{Z_\mathrm{p} e_0^2}{8\pi\epsilon_0 pv}\right)^2 \left(16\frac{p^4}{q^4}\right)\cdot\frac{\pi}{p^2} = \frac{1}{4\pi}\left(\frac{Z_\mathrm{p} e_0^2}{\epsilon_0 v q^2}\right)^2. \tag{2.3}$$

Next we find the relationship between q^2 and kinetic energy. Now, T, by absolute value, is both the kinetic energy of the electron after the collision (assuming the electron was originally at rest) and the energy lost by the ionizing particle during the collision.

Next, what is q^2 in the rest system of the electron? In this system the electron's initial momentum is

$$p^\mu = \left(\frac{m_\mathrm{e}c^2}{c}, \mathbf{0}\right) \qquad \text{and} \qquad p'^\mu = \left(\frac{E}{c}, \mathbf{p}'\right) = \left(\frac{m_\mathrm{e}c^2 + T}{c}, \mathbf{p}'\right)$$

while the difference q is

$$q^\mu \equiv p'^\mu - p^\mu = \left(\frac{T}{c}, \mathbf{p}'\right).$$

Meanwhile, the invariant quantity $(q^\mu)^2 = (p'^\mu - p^\mu)^2$ comes out to

$$(q^\mu)^2 = q^\mu q_\mu = (p')^2 - \left(\frac{T}{c}\right)^2. \tag{2.4}$$

Next, we write $p' = |\mathbf{p}'|$ in terms of kinetic energy. We have

$$(m_\mathrm{e}c^2 + T)^2 = E^2 = m_\mathrm{e}^2c^4 + (p')^2c^2$$

where E is the electron's total relativistic energy and the last line uses the general relationship $E^2 = m^2c^4 + p^2c^2$. We then solve this equality for $(p')^2$ to get

$$p'^2 = \frac{T^2}{c^2} + 2m_\mathrm{e}T.$$

In this case we can solve Equation 2.4 for the invariant quantity $(q^\mu)^2$, which is

$$(q^\mu)^2 = \left(\frac{T^2}{c^2} + 2m_\mathrm{e}T\right) - \frac{T^2}{c^2} = 2m_\mathrm{e}T.$$

We then subsitute in the expression for $q^2 = 2m_\mathrm{e}T$ into Equation 2.3 and get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{\mathrm{d}\sigma}{\mathrm{d}[2m_\mathrm{e}T]} = \frac{\mathrm{d}\sigma}{2m_\mathrm{e}\,\mathrm{d}T} = \frac{1}{4\pi}\left(\frac{Z_\mathrm{p}e_0^2}{\epsilon_0 v(2m_\mathrm{e}T)}\right)^2.$$

Solving the equation for $\frac{\mathrm{d}\sigma}{\mathrm{d}T}$ results in

$$\frac{\mathrm{d}\sigma}{\mathrm{d}T} = \frac{1}{8\pi\epsilon_0^2}\frac{Z_\mathrm{p}^2 e_0^4}{m_\mathrm{e}c^2\beta^2T^2},$$

where we have used $v = c\beta$.

Energy Losses

We return to Equation 2.1 at the beginning of the derivation and substitute in $\frac{\mathrm{d}\sigma}{\mathrm{d}T}$, producing

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = n_\mathrm{a} Z_\mathrm{m} \int_{T_\mathrm{min}}^{T_\mathrm{max}} T \frac{\mathrm{d}\sigma}{\mathrm{d}T}\,\mathrm{d}T = n_\mathrm{a} Z_\mathrm{m} \int_{T_\mathrm{min}}^{T_\mathrm{max}} T \frac{1}{8\pi\epsilon_0^2} \frac{Z_\mathrm{p}^2 e_0^4}{m_\mathrm{e} c^2 \beta^2 T^2}\,\mathrm{d}T$$

This comes out to

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{n_\mathrm{a} Z_\mathrm{m}}{8\pi\epsilon_0^2} \frac{Z_\mathrm{p}^2 e_0^4}{m_\mathrm{e} c^2 \beta^2} \ln \frac{T_\mathrm{max}}{T_\mathrm{min}} \tag{2.5}$$

For review, n_a is the number density of atoms in the material through which the ionizing particle travels and Z_m is the material's atomic number.

We know the maximum energy loss T_max from Equation 1.6 in the previous exercise set. Assuming the ionizing particle is much heavier than an electron, T_max is

$$T_\mathrm{max} = 2 m_\mathrm{e} c^2 \gamma^2 \beta^2$$

A formal expression for minimum energy T_min derived from fundamental principles is more tricky. It is conventional avoid this and simply parameterize T_min as

$$T_\mathrm{min} = I = Z_\mathrm{m} I_0$$

where I is an average effective binding energy (also called mean excitation potential) of the electrons in the material to their parent atoms, and depends on the material. A typical value is of the order $I_0 \sim 10\,\mathrm{eV}$.

The Bethe-Bloch Formula

After substituting T_min and T_max into Equation 2.5, the expression for the ionizing particle's energy losses reads

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{n_\mathrm{a} Z_\mathrm{m}}{8\pi\epsilon_0^2} \frac{Z_\mathrm{p}^2 e_0^4}{m_\mathrm{e} c^2 \beta^2} \ln \left(\frac{2 m_\mathrm{e} c^2 \gamma^2 \beta^2}{Z_\mathrm{m} I_0} \right).$$

We then note that $n_\mathrm{a} Z_\mathrm{m}$ is the number density n_e of electrons in the material and then rearrange the factor 4π slightly to get the conventional formulation

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{4\pi}{m_\mathrm{e} c^2} \cdot \frac{n_\mathrm{e} Z_\mathrm{p}^2}{\beta^2} \cdot \left(\frac{e_0^2}{4\pi\epsilon_0} \right)^2 \ln \left(\frac{2 m_\mathrm{e} c^2 \gamma^2 \beta^2}{Z_\mathrm{m} I_0} \right).$$

Finally, without derivation, corrections for the electron's spin produce

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{4\pi}{m_\mathrm{e} c^2} \cdot \frac{n_\mathrm{e} Z_\mathrm{p}^2}{\beta^2} \cdot \left(\frac{e_0^2}{4\pi\epsilon_0} \right)^2 \left[\ln \left(\frac{2 m_\mathrm{e} c^2 \gamma^2 \beta^2}{Z_\mathrm{m} I_0} \right) - \beta^2 \right].$$

This is the *Bethe-Bloch equation.* In practice, we will often use the equivalent formulation

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = K \cdot \frac{\rho Z_\mathrm{m}}{A} \cdot \frac{Z_\mathrm{p}^2}{\beta^2} \left[\ln \left(\frac{2 m_\mathrm{e} c^2 \beta^2 \gamma^2}{Z_\mathrm{m} I_0} \right) - \beta^2 \right], \tag{2.6}$$

where $K = 0.3\,\mathrm{MeV\,g^{-1}\,cm^2}$.

A few comments:

- The formula holds for values of $\beta\gamma$ in the (approximate) range $\beta\gamma \in (0.05, 10^3)$.
- The minimum occurs in the range $\beta\gamma \approx 3 - 4$.
- Particle with energy in the minimum are called minimum ionizing particles.
- The $1/\beta^2$ holds in the low speed regime and the logarithm holds in the high speed regime.

2.2 Numerical Values of Ionizing Losses in Argon

Find the ionizing energy losses per unit distance traveled through argon for a proton, kaon, and muon, all with momentum $pc = 1\,\mathrm{GeV}$. Argon's atomic number, mass number, and density are $Z = 18$, $A = 40$, and $\rho = 4 \cdot 10^{-3}\,\mathrm{g\,cm^{-3}}$, respectively.

This problem is essentially an exercise in using the Bethe-Bloch formula in Equation 2.7 with given experimental data. For review, the Equation 2.7 formulation of the Bethe-Bloch equation reads

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = K \cdot \frac{\rho Z_\mathrm{m}}{A} \cdot \frac{Z_\mathrm{p}^2}{\beta^2}\left[\ln\left(\frac{2m_\mathrm{e}c^2\beta^2\gamma^2}{Z_\mathrm{m}I_0}\right) - \beta^2\right], \tag{2.7}$$

where $K = 0.3\,\mathrm{MeV\,g^{-1}\,cm^2}$ and $I_0 \sim 10\,\mathrm{eV}$. We aim to find expressions for each of the terms in this equation using the given experimental data.

We begin by calculating the values of β and γ for each particle using the equation

$$\gamma\beta = \frac{pc}{mc^2}$$

With the product $\gamma\beta$ known, we find γ using

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \implies \gamma^2 = 1 + (\beta\gamma)^2 \qquad \text{and} \qquad \beta^2 = \frac{\gamma^2 - 1}{\gamma^2}.$$

The proton, kaon and muon rest energies are approximately $m_\mathrm{p}c^2 \approx 1\,\mathrm{GeV}$, $m_K c^2 \approx 500\,\mathrm{MeV}$ and $m_\mu c^2 \approx 110\,\mathrm{MeV}$, respectively. The corresponding relevant kinematic quantities needed for the Bethe-Bloch equation are then

Quantity	p	K	μ
$\beta\gamma$	1	2	10
γ^2	2	5	101
β^2	1/2	4/5	~ 1
$\frac{2m_\mathrm{e}c^2\beta^2\gamma^2}{Z_\mathrm{Ar}I_0}$	$5 \cdot 10^3$	$20 \cdot 10^3$	$500 \cdot 10^3$

Finally, the factor $K\frac{\rho Z_\mathrm{m}}{A}$ comes out to

$$K \cdot \frac{\rho Z_\mathrm{m}}{A} = 0.3\,\mathrm{MeV\,g^{-1}\,cm^2} \cdot \frac{(4 \cdot 10^{-3}\,\mathrm{g\,cm^{-3}}) \cdot 18}{40} \approx 5.4 \cdot 10^{-4}\,\mathrm{MeV\,cm^{-1}}$$

The corresponding ionizing energy losses $\frac{\mathrm{d}E}{\mathrm{d}x}$ at momentum $pc = 1\,\mathrm{GeV}$ then come out to

$$\frac{\mathrm{d}E_\mathrm{p}}{\mathrm{d}x} \approx 8\,\mathrm{GeV\,cm^{-1}}, \qquad \frac{\mathrm{d}E_K}{\mathrm{d}x} \approx 6\,\mathrm{GeV\,cm^{-1}}, \qquad \frac{\mathrm{d}E_\mu}{\mathrm{d}x} \approx 6\,\mathrm{GeV\,cm^{-1}}$$

for the proton, kaon, and muon respectively. Finally, without derivation, we quote that a similar experiment with $pc = 500\,\mathrm{MeV}$ particles would result in

$$\frac{\mathrm{d}E_\mathrm{p}}{\mathrm{d}x} \approx 17\,\mathrm{GeV\,cm^{-1}}, \qquad \frac{\mathrm{d}E_K}{\mathrm{d}x} \approx 8\,\mathrm{GeV\,cm^{-1}}, \qquad \frac{\mathrm{d}E_\mu}{\mathrm{d}x} \approx 5.6\,\mathrm{GeV\,cm^{-1}}.$$

3 Third Exercise Set

3.1 Proton Therapy (Ionizing Energy Losses in Water)

Proton therapy uses protons with energy $T = 250\,\mathrm{MeV}$. Estimate the distribution of ionizing energy losses $\frac{\mathrm{d}E}{\mathrm{d}x}$ with respect to the distance x a proton has traveled through the body. For easier analysis, assume the body is made entirely of water.

The problem essentially reduces to computed the ionizing energy losses of a proton in water. Our first step is to generalize the Bethe-Bloch equation to apply to a diatomic molecule. The appropriate generalization is

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = \sum_i w_i \frac{\mathrm{d}E}{\mathrm{d}x}\bigg|_i,$$

where w_i is the mass weight of each element, given by

$$w_i = \frac{a_i A_i}{\sum_j a_j A_j}$$

where a_i is the number of type i atoms in the molecule (e.g. $a_\mathrm{H} = 2$ and $a_\mathrm{O} = 1$ for H_2O), while A_i is the i-th element's atomic number.

More so, we must make the following generalizations:

- replace Z with Z_eff, where $Z_\mathrm{eff} = \sum_i a_i Z_i$
- replace A with A_eff according to $A \to A_\mathrm{eff} = \sum_i a_i A_i$, and
- replace I with I_eff according to $\ln I \to \ln I_\mathrm{eff} = \sum_i \frac{a_i Z_i \ln I_i}{Z_\mathrm{eff}}$.

For water we have $A_\mathrm{H} = 1$ and $Z_\mathrm{H} = 1$ for hydrogen and $A_\mathrm{O} = 16$ and $Z_\mathrm{O} = 8$ for oxygen. We then have

$$Z_\mathrm{eff} = 2 \cdot 1 + 1 \cdot 8 = 10 \qquad \text{and} \qquad A_\mathrm{eff} = 2 \cdot 1 + 1 \cdot 16 = 18.$$

Next, we find I_eff; this is simply quoted as the tabulated value $I_\mathrm{eff} = 75\,\mathrm{eV}$, since water is such a common substance.

Next, we pause for an intermediate calculation of kinematic quantities in the Bethe-Bloch equation. We compute the proton's γ factor according to

$$E = m_\mathrm{p}c^2\gamma = m_\mathrm{p}c^2 + T \implies \gamma = 1 + \frac{T}{m_\mathrm{p}c^2} = 1 + \frac{250\,\mathrm{MeV}}{1000\,\mathrm{MeV}} \approx \frac{5}{4}.$$

With γ known, we find the product $\beta\gamma$ according to

$$\beta^2\gamma^2 = \gamma^2 - 1 = \frac{25}{16} - 1 = \frac{9}{16} \implies \beta\gamma = \frac{3}{4}.$$

Using the known values of γ and $\beta\gamma$, the proton's β factor is

$$\beta^2 = \frac{(\beta\gamma)^2}{\gamma^2} = \frac{9}{25} \implies \beta = \frac{3}{5}.$$

We are interested in the proton's range (i.e. distance traveled through the body), so we want to write energy losses as a function of distance x, not as a function of β.

We begin with the small β^2 approximation $-\frac{\mathrm{d}E}{\mathrm{d}x} \sim \beta^{-2} \sim T^{-1}$; in our case the ionizing proton has $\beta\gamma \approx 0.75 < 1$, which, to a first approximation, is good enough to assume the β^{-2} regime. We thus write

$$-\frac{\mathrm{d}E}{\mathrm{d}x} \propto \frac{1}{\beta^2} \propto \frac{1}{T}.$$

Since the only change in proton energy comes from kinetic energy, we can write

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{\mathrm{d}T}{\mathrm{d}x},$$

which, using $-\frac{\mathrm{d}E}{\mathrm{d}x} \propto \frac{1}{T}$, allows us to write

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{\mathrm{d}T}{\mathrm{d}x} \propto -\frac{1}{T} \quad \text{or} \quad \frac{\mathrm{d}T}{\mathrm{d}x} = -\frac{k}{T}. \tag{3.1}$$

We find the constant k from any point on the $\frac{\mathrm{d}E}{\mathrm{d}x}$ curve in the small β regime, for which the approximation $-\frac{\mathrm{d}E}{\mathrm{d}x} \propto T^{-1}$ is valid. For example, we could find k from

$$k = -T_0 \cdot \frac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{T=T_0}, \tag{3.2}$$

where $T_0 = 250\,\mathrm{MeV}$ is the known proton energy given in the problem instructions.

We now compute k. As an intermediate step, we first find $\frac{\mathrm{d}T}{\mathrm{d}x}$ from the Bethe-Bloch equation. This is

$$-\frac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{T=T_0} = -\frac{\mathrm{d}E}{\mathrm{d}x}\bigg|_{T=T_0} = K\frac{\rho Z_{\mathrm{eff}}}{A_{\mathrm{eff}}}\frac{1^2}{\beta^2}\left[\ln\left(2\frac{\gamma^2\beta^2 m_{\mathrm{e}}c^2}{I_{\mathrm{eff}}}\right) - \beta^2\right],$$

where we have used $Z_{\mathrm{p}} = 1$ for a proton. This comes out to (confirm)

$$-\frac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{T=T_0} \approx 3.6\,\mathrm{MeV\,cm^{-1}}.$$

We can then find k using Equation 3.2 according to

$$k = -T_0\frac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{T=T_0} = 250\,\mathrm{MeV} \cdot 3.6\,\mathrm{MeV\,cm^{-1}} = 900\,\mathrm{MeV^2\,cm^{-1}}.$$

Having computed k, we now return to Equation 3.1 and integrate to get

$$\frac{\mathrm{d}T}{\mathrm{d}x} = -\frac{k}{T} \implies \int_{T_0}^{T} T' \,\mathrm{d}T' = \frac{1}{2}(T^2 - T_0^2) = -k \int_0^x \mathrm{d}x' = -kx.$$

We then solve for $T(x)$ and get

$$T(x) = \sqrt{T_0^2 - 2kx}. \tag{3.3}$$

By differentiating the above expression for $T(x)$, the proton's ionizing losses as a function of distance x traveled through the body are

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{\mathrm{d}T}{\mathrm{d}x} = -\frac{k}{\sqrt{T_0^2 - 2kx}}. \tag{3.4}$$

The kinetic energy at the hypothetical range $x = R$ is by definition equal to zero. From Equation 3.3, we see $T = 0$ when

$$T_0^2 = 2kR \implies R = \frac{T_0^2}{2k} = \frac{(250\,\mathrm{MeV})^2}{2 \cdot (900\,\mathrm{MeV}^2\,\mathrm{cm}^{-1})} \approx 35\,\mathrm{cm}.$$

In other words, we can estimate the range of penetration as long as we know the initial proton energy and the contant k.

Note that the Bethe-Bloch equation doesn't apply for very small β, which is why, in experiment, we don't observe divergence of $\frac{\mathrm{d}E}{\mathrm{d}x}$ at $x = R$ as Equation 3.4 predicts. This disagreement with experiment is expected; Equation 3.4 was derived from the Bethe-Bloch formula, which wouldn't apply as $x \to R$, and so the equation's prediction as $x \to R$ is non-physical.

Note also that $\frac{\mathrm{d}E}{\mathrm{d}x}$ is called a Bragg curve, and the peak is the Bragg peak. A typical Bragg peak for a 250 MeV proton in water might occur at $\frac{\mathrm{d}E}{\mathrm{d}x} \approx 30\,\mathrm{MeV\,cm^{-1}}$.

Finally, for orientation, we compute the position $x_{1/2}$ at which the particle gives up half of its initial kinetic energy. With reference to Equation 3.3, finding $x_{1/2}$ involves solving the equation

$$T(x_{1/2}) = \frac{T_0}{2} = \sqrt{T_0^2 - 2kx_{1/2}}.$$

After squaring both sides and rearranging, we find

$$\frac{T_0^2}{4} = T_0^2 - 2kx_{1/2} \implies x_{1/2} = \frac{3}{4}\frac{T_0^2}{2k} = \frac{3}{4}R.$$

In other words, the particle gives up half of its total kinetic energy only in the final fourth of its path through the water (body). This is much more desirable than exponentially-attenuating gamma radiation, which deposits most its energy near the body's surface (and far away from the tumor inside the body it is supposed to treat).

3.2 Theory: Photon Interaction with Matter

We consider three important contributions to the total cross section for photon interaction with matter: (i) the photoelectric effect, (ii) Compton scattering, and (iii) pair production. We thus write the total cross section for photon interaction with matter as

$$\sigma_\gamma = \sigma_{\mathrm{pe}} + \sigma_{\mathrm{c}} \cdot Z + \sigma_{\mathrm{pair}}.$$

We now briefly discuss each contribution in turn.

(i) The photoelectric effect tends to domminate for photons with energy $E_\gamma \lesssim 100\,\mathrm{keV}$. The photoelectric cross section may be roughly approximated as

$$\sigma_{\mathrm{pe}} \approx \frac{Z^n}{E_\gamma^{7/2}},$$

where the exponent n generally falls in the range four to five. Note that this expression neglects peaks at K and L edges.

(ii) The cross section for Compton scattering is roughly (for $E_\gamma \ll m_{\mathrm{e}}c^2$) given by

$$\sigma_{\mathrm{C}} \approx \frac{8\pi}{3} r_{\mathrm{e}}^2,$$

where r_{e} is the classical electron radius, given by

$$r_{\mathrm{e}} = \frac{1}{4\pi\epsilon_0} \frac{e_0^2}{m_{\mathrm{e}}c^2}$$

(iii) Pair production occurs for high-energy photons, *in the presence of atomic nuclei*, which then decay into an electron-positron pair. The creaton of these two particles immediately implies an energy threshold $E_\gamma > 2m_{\mathrm{e}}c^2$, i.e. the photon must have energy larger that the sum of the electron and positron rest masses.

For orientation, we quote a few representative cross sections below

Scenario	σ_{C} [b]	σ_{pe} [b]	σ_{pair} [b]
$E_\gamma = 1\,\mathrm{MeV}$ in aluminum	3	10^{-3}	-
$E_\gamma = 0.1\,\mathrm{MeV}$ in hydrogen	0.5	10^{-4}	-
$E_\gamma = 10\,\mathrm{MeV}$ in lead	4	-	12

The attenuation constant μ for photon interaction in a material of density ρ and molar mass M_{m} with photon scattering cross section σ_γ is

$$\mu = \frac{\rho N_A}{M_{\mathrm{m}}} \sigma_\gamma = n_{\mathrm{a}} \sigma_\gamma, \tag{3.5}$$

where n_{a} is the number density of atomic nuclei in the material. We can also define the associated characteristic attenuation length

$$\lambda = \frac{1}{\mu},$$

which, physically, represents the mean path of a photon travelling through material.

The intensity of a photon beam falls exponentially with distance traveled through matter. For a photon flux density j_0 incident on a material, the flux $j(x)$ a distance x into the material is

$$j(x) = j_0 e^{-\mu x},$$

Finally, the photon interaction "half-distance" $d_{1/2}$ is the distance at which

$$\frac{j(d_{1/2})}{j_0} = \frac{1}{2} = e^{-d_{1/2}\mu} \implies d_{1/2} = \frac{\ln 2}{\mu}.$$

3.3 Compton Scattering Attenuation in Sodium Iodide

Compute the attenuation coefficient μ for Compton scattering and the photoelectric effect in sodium iodide (NaI), which has density $\rho = 3.7\,\mathrm{g\,cm^{-3}}$. Then determine the attenuation constant μ_{tot} associated with the total photon interaction cross section, and the fraction of Compton interactions, for photons with $E_\gamma = 140\,\mathrm{keV}$. The following data is given:

	Z	A	μ_C/ρ [$\mathrm{g^{-1}\,cm^2}$]	μ_{pe}/ρ [$\mathrm{g^{-1}\,cm^2}$]
Na	11	23	0.13	$3 \cdot 10^{-3}$
I	53	127	0.1	0.66

We begin by computing the weights of each element in sodium iodide:

$$w_{\mathrm{Na}} = \frac{A_{\mathrm{Na}}}{A_{\mathrm{Na}} + A_{\mathrm{I}}} = \frac{23}{23 + 127} \approx 0.15 \quad \text{and} \quad w_{\mathrm{I}} = \frac{A_{\mathrm{I}}}{A_{\mathrm{Na}} + A_{\mathrm{I}}} = \frac{127}{23 + 127} \approx 0.85.$$

We then find the total density-normalized attenuation coefficient for Compton scattering using a weighted combination of the individual atom's attenuation coefficients:

$$\begin{aligned} \frac{\mu_C^{\mathrm{NaI}}}{\rho} &= w_{\mathrm{Na}} \cdot \frac{\mu_C^{\mathrm{Na}}}{\rho} + w_{\mathrm{I}} \cdot \frac{\mu_C^{\mathrm{I}}}{\rho} = 0.15 \cdot 0.13\,\mathrm{g^{-1}\,cm^2} + 0.85 \cdot 0.1\,\mathrm{g^{-1}\,cm^2} \\ &\approx 0.1\,\mathrm{cm^2\,g^{-1}}. \end{aligned} \tag{3.6}$$

The Compton scattering attenuation coefficient is thus

$$\mu_C^{\mathrm{NaI}} = \rho \cdot \left(\frac{\mu_C^{\mathrm{NaI}}}{\rho}\right) = 3.7\,\mathrm{g\,cm^{-3}} \cdot 0.1\,\mathrm{cm^2\,g^{-1}} \approx 0.4\,\mathrm{cm^{-1}}$$

The proceedure for the photoelectric effect is analogous; we simply replace the μ_C terms in Equation 3.6 with μ_{pe} terms. The calculation reads

$$\begin{aligned} \frac{\mu_{pe}^{\mathrm{NaI}}}{\rho} &= w_{\mathrm{Na}} \cdot \frac{\mu_{pe}^{\mathrm{Na}}}{\rho} + w_{\mathrm{I}} \cdot \frac{\mu_{pe}^{\mathrm{I}}}{\rho} = 0.15 \cdot 3 \cdot 10^{-3}\,\mathrm{g^{-1}\,cm^2} + 0.85 \cdot 0.66\,\mathrm{g^{-1}\,cm^2} \\ &\approx 0.56\,\mathrm{cm^2\,g^{-1}}. \end{aligned}$$

The photoelectric attenuation coefficient is thus

$$\mu_{pe}^{\mathrm{NaI}} = \rho \cdot \left(\frac{\mu_{pe}^{\mathrm{NaI}}}{\rho}\right) = 3.7\,\mathrm{g\,cm^{-3}} \cdot 0.56\,\mathrm{cm^2\,g^{-1}} \approx 2.1\,\mathrm{cm^{-1}}.$$

The total attenuation coefficient is simply the sum of the two contributions:

$$\mu_{\text{tot}} = \mu_{\text{C}} + \mu_{\text{pe}} = 2.5\,\text{cm}^{-1}$$

The fraction of Compton scattering interactions is then

$$\frac{\mu_{\text{C}}}{\mu_{\text{tot}}} = \frac{0.4}{2.5} = 0.16,$$

meaning about 16% of incident photons interact with NaI via the Compton effect.

4 Fourth Exercise Set

4.1 Compton Scattering

Compute the energy of a 1 MeV *photon after the photon Compton scatters from an electron at an angle of* $90°$ *relative to the direction of incidence.*

We first define the relevant kinematic quantities:

- $\mathbf{k}$ and E_γ denote the photon's momentum and energy before scattering
- $\mathbf{k}'$ and E'_γ denote the photon's momentum and energy after scattering
- $\mathbf{p}$ and T denote the electron's momentum and (kinetic) energy after scattering.

We begin with conservation of momentum, which requires

$$\mathbf{p} = \mathbf{k} - \mathbf{k}',$$

We then square to get the scalar equation

$$p^2 = k^2 + k'^2 - 2k'k\cos\theta, \tag{4.1}$$

where θ is the angle between the photon's initial and scattered momentum. For our specific problem, this angle is $\theta = 90°$, but will solve the problem with a general θ.

Simultaneously, we apply conservation of energy, which requires

$$E_\gamma + m_e c^2 + E'_\gamma + m_e c^2 + T \implies T = E_\gamma - E'_\gamma,$$

Next, in terms of kinetic energy, the electron's momentum after scattering reads

$$(T + m_e c^2) = m_e c^4 + p^2c^2 \implies p^2c^2 = T^2 + 2m_e c^2 T$$

We then substitute in $T = E_\gamma - E'_\gamma$ to get

$$p^2c^2 = T^2 + 2m_e c^2 T = (E_\gamma - E'_\gamma)^2 + 2m_e c^2 (E_\gamma - E'_\gamma)^2. \tag{4.2}$$

For a photon, which has zero mass, the relationship between momentum and energy is simply $ck = E_\gamma$ and $ck' = E'_\gamma$, in terms of which Equation 4.1 reads

$$p^2c^2 = E_\gamma^2 + E_\gamma'^2 - 2E'_\gamma E_\gamma \cos\theta, \tag{4.3}$$

We then equate Equations 4.3 and 4.2 to get

$$E_\gamma^2 + E_\gamma'^2 - 2E_\gamma E'_\gamma \cos\theta = (E_\gamma - E'_\gamma)^2 + 2m_e c^2 (E_\gamma - E'_\gamma),$$

and then multiply out and cancel $E_\gamma^2 + E_\gamma'^2$ from both sides, which gives

$$-2E_\gamma E'_\gamma \cos\theta = -2E_\gamma E'_\gamma + 2m_e c^2 (E_\gamma - E'_\gamma)$$

We then rearrange and solve for E'_γ to get

$$E'_\gamma(-2E_\gamma \cos\theta + 2E_\gamma + 2m_e c^2) = 2m_e c^2 E_\gamma \implies E'_\gamma = \frac{m_e c^2 E_\gamma}{E_\gamma + m_e c^2 - E_\gamma \cos\theta}.$$

A few more steps of algebra then gives the final expression

$$\frac{E'_\gamma}{E_\gamma} = \frac{1}{1+\alpha(1-\cos\theta)}, \quad \text{where} \;\; \alpha \equiv \frac{E_\gamma}{m_\mathrm{e}c^2}. \tag{4.4}$$

Concrete Values for a 1 MeV Photon

For this problem's concrete case with $E_\gamma = 1\,\mathrm{MeV}$ and $\theta = 90°$, we have

$$\alpha = \frac{1\,\mathrm{MeV}}{0.51\,\mathrm{MeV}} \approx 2 \quad \text{and} \quad \cos\theta = 0 \implies E'_\gamma = \frac{E_\gamma}{1+2} \approx 0.33\,\mathrm{MeV}.$$

Maximum energy transfer to the electron occurs when $\theta = \pi$ (a head-on collision with the photon "rebounding" (backscattering) along the direction of incidence) and thus $\cos\theta = -1$, in which case the final photon energy, assuming $E_\gamma = 1\,\mathrm{MeV}$, is

$$E'_\mathrm{min} = \frac{E_\gamma}{1+2\alpha} \approx 0.2\,\mathrm{MeV}$$

The corresponding maximum energy transfer to the electron is

$$T_\mathrm{max} = E_\gamma - E'_\mathrm{min} = 0.8\,\mathrm{MeV}.$$

The energy T_max is called the Compton edge, and is usually visible in the spectra of photon scattering processes.

We conclude by noting that we can write T_max in the form

$$T_\mathrm{max} = \frac{2\alpha E_\gamma}{1+2\alpha}.$$

Interpretation: T_max/E_γ is bounded above at one. For large α, the electron can gain nearly the entire incident photon energy.

Note that in the large α regime, it becomes nearly impossible to separate the photoelectric effect from Compton scattering, because in the photeelectric effect, the photon also looses its entire energy. In both cases entire photon energy is lost and transferred to an electron.

4.2 Theory: Compton Scattering Formulae

The (Klein-Nishina) differential scattering cross section for the Compton effect (i.e. photon scattering from an electron) is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{r_\mathrm{e}^2}{2}\left(\frac{E'_\gamma}{E_\gamma}\right)^2 \left[\frac{E'_\gamma}{E_\gamma} + \frac{E_\gamma}{E'_\gamma} - \sin^2\theta\right],$$

where $r_\mathrm{e} = \frac{1}{4\pi\epsilon_0}\frac{e_0}{m_\mathrm{e}c^2} \sim 2.8\,\mathrm{fm}$ is the classical electron radius and the ratio E'_γ/E_γ between scattered and initial photon energies is given in Equation 4.4, which for review reads

$$\frac{E'_\gamma}{E_\gamma} = \frac{1}{1+\alpha(1-\cos\theta)}, \quad \text{where} \;\; \alpha \equiv \frac{E_\gamma}{m_\mathrm{e}c^2}.$$

The total cross section for Compton scattering is approximately

$$\sigma_{\mathrm{C}} \approx \frac{8\pi r_{\mathrm{e}}^2}{3}\left[\frac{1-2\alpha+1.2\alpha^2}{(1+2\alpha)^2}\right]. \tag{4.5}$$

Finally, we aim to find the electron's kinetic energy spectrum $\frac{\mathrm{d}\sigma_{\mathrm{c}}}{\mathrm{d}T}$ after Compton scattering. The derivation would use the change of variables

$$\frac{\mathrm{d}\sigma_{\mathrm{c}}}{\mathrm{d}\Omega} \to \frac{\mathrm{d}\sigma_{\mathrm{c}}}{\mathrm{d}E'_\gamma} \to \frac{\mathrm{d}\sigma_{\mathrm{c}}}{\mathrm{d}T}$$

using $T = E_\gamma - E'_\gamma$. However, se simply quote the result which is:

$$\frac{\mathrm{d}\sigma_{\mathrm{C}}}{\mathrm{d}T} = \frac{\pi r_0^2}{m_{\mathrm{e}}c^2\alpha^2}\left[2 + \frac{s^2}{\alpha^2(1-s)^2} + \frac{s}{1-s}\left(s - \frac{2}{\alpha}\right)\right],$$

where $s \equiv T/E_\gamma$ is the ratio of the electron kinetic energy after scattering to the incident photon energy.

4.3 Compton Scattering in Sodium Iodide

Compute the mean free path for $1\,\mathrm{MeV}$ *photons in sodium iodide (density* $\rho = 3.7\,\mathrm{g\,cm^{-3}}$*) for the Compton effect. The atomic number and mass number in sodium and iodine are* $Z_{\mathrm{Na}} = 11$ *and* $A_{\mathrm{Na}} = 23$*; and* $Z_{\mathrm{I}} = 53$ *and* $A_{\mathrm{I}} = 127$*, respectively.*

We first find sodium iodide's effective atomic number with the weighted sum

$$Z_{\mathrm{NaI}} = \sum_{i\in\{\mathrm{Na,I}\}} m_i Z_i = \frac{23}{23+127}\cdot 11 + \frac{127}{127+23}\cdot 53 \approx 46.6.$$

For our problem, with $E_\gamma = 1\,\mathrm{MeV}$, we have $\alpha = E_\gamma/(m_{\mathrm{e}}c^2) \approx 2$. The corresponding scattering cross section for photons from a *single* electron, using Equation 4.5, is

$$\sigma_{\mathrm{C}}^{(1)} = \frac{8\pi r_0^2}{3}\left[\frac{1-2\alpha+1.2\alpha^2}{(1+2\alpha)^2}\right] \approx 0.26\,\mathrm{b}.$$

The corresponding compton scattering cross from a sodium iodide molecule (which, naturally, has many electrons) is found by multiplying the single electron result by sodium iodide's effective atomic number:

$$\sigma_{\mathrm{C}}(1\,\mathrm{MeV};\mathrm{NaI}) = Z_{\mathrm{NaI}}\cdot\sigma_{\mathrm{C}}^{(1)} \approx 46.6\cdot 0.26\,\mathrm{b} \approx 12\,\mathrm{b}.$$

Finally, we find mean free path λ from the attenuation coefficient $\mu = 1/\lambda$ in Equation 3.5, which for review reads

$$\frac{1}{\lambda} = \mu = n_{\mathrm{a}}\sigma_\gamma = \frac{\rho N_A}{M_{\mathrm{m}}}\sigma_\gamma,$$

where M_{m} is the molar mass of the material through which the photon travels, in our case sodium iodide, and σ_γ is the total photon cross section. Assuming all photon interactions come from Compton scattering, in which case $\sigma_\gamma = \sigma_C$, and rearranging, we have

$$\lambda = \frac{M_{\mathrm{NaI}}}{\rho N_A}\frac{1}{\sigma_C}.$$

Using the approximation $M_{\mathrm{NaI}} \approx A_{\mathrm{Na}} \cdot M_u + A_I \cdot M_u = 23\,\mathrm{g\,mol^{-1}} + 127\,\mathrm{g\,mol^{-1}} = 150\,\mathrm{g\,mol^{-1}}$ (where $M_u = 1\,\mathrm{g\,mol^{-1}}$ is the molar mass constant), we have

$$\lambda = \frac{150\,\mathrm{g\,mol^{-1}}}{(3.7\,\mathrm{g\,cm^{-3}}) \cdot (6 \cdot 10^{23}\,\mathrm{mol^{-1}})} \cdot \frac{1}{12\,\mathrm{b}} \approx 5.6\,\mathrm{cm}.$$

4.4 Theory: Signal in a Rectangular Ionization Cell

For orientation: a rectangular ionization cell is a rectangular, parallel-plate chamber filled with gas (typically a noble gas such as argon mixed with an organic gas such as carbon dioxide).

The chamber plates are held at a kilovolt-order potential difference, forming a capacitor. An ionizing particle passing through the cell ionizes the gas in cell; the positive and negative ions are accelerated to the negative and positive capacitor faces by the potential gradient. This charge pulse, via appropriate read-out electronics, results in temporary drop in voltage indicating the presence of the initial ionizing particle.

Important: the potential difference must be large enough to avoid recombination of ions before they reach the capacitor edges. A typical electric field would be of the order 100 to $1000\,\mathrm{V\,cm^{-1}}$.

In a parallel-plate capacitor with plate spacing d, potential difference U and electric field $\mathbf{E} = (U/d)\,\hat{\mathbf{n}}$ (where $\hat{\mathbf{n}}$ is the normal to the plate faces), the work $\mathrm{d}W$ done on moving charge q through a distance $\mathrm{d}x$ (assuming $\mathrm{d}\mathbf{x} \parallel \mathbf{E}$) is

$$\mathrm{d}W = q\mathbf{E} \cdot \mathrm{d}\mathbf{x} = qE\,\mathrm{d}x = \frac{qU}{d}\,\mathrm{d}x.$$

The corresponding change in the capacitor's energy (using W_C to denote capacitor energy, to avoid conflict with electric field) is

$$W_C = \frac{1}{2}CU^2 \implies \mathrm{d}W_C = CU\,\mathrm{d}U.$$

Conservation of energy requires that the work done by the electric field in moving the charge equals the change in the capacitor's energy. In symbols:

$$\mathrm{d}W = \mathrm{d}W_C \implies q\frac{U}{d}\,\mathrm{d}x = CU\,\mathrm{d}U \implies \mathrm{d}U = \frac{q}{C}\frac{\mathrm{d}x}{d}. \tag{4.6}$$

In other words, as a charge q moves through the ionizing cell by a distance $\mathrm{d}x$, the voltage on the cell's plates changes by $\mathrm{d}U$.

To find the voltage signal's dynamics $U(t)$, we must express $\mathrm{d}U$ in terms of time $\mathrm{d}t$ instead of position $\mathrm{d}x$. We can approximate the motion of ions through the gas chamber as constant-velocity motion at a drift speed v_d. (Technically, on a microscopic scale, ions accelerate in the electric field, collide with gas atoms and slow, accelerate again, collide again, etc...)

We model the drift velocity of positive and negative ions as

$$v_{\mathrm{d}}^{(\pm)} = \mu_{\pm}\frac{E}{p},$$

where p is pressure in the detector, E is electric field, and $\mu_{\pm}$ is hole or electron mobility. Typical electron drift velocities are in the range $v_{\mathrm{d}}^{(-)} \sim 10^4$ to $10^5\,\mathrm{m\,s^{-1}}$, while ions move considerably slower, at $v_{\mathrm{d}}^{(+)} \sim 10\,\mathrm{m\,s^{-1}}$.

We convert from position to time using drift velocity according to

$$v_{\mathrm{d}} = \frac{\mathrm{d}x}{\mathrm{d}t} \implies \mathrm{d}x = v_{\mathrm{d}}\,\mathrm{d}t = \frac{\mu E}{p}\,\mathrm{d}t.$$

The voltage drop on the detector plates is then

$$\mathrm{d}U = \frac{q}{C}\frac{\mathrm{d}x}{d} = \frac{q}{Cd}\frac{\mu E}{p}\,\mathrm{d}t$$

A parallel-plate capacitor's electric field is constant, so we can easily integrate to get

$$\int_{U_0}^{U(t)} \mathrm{d}U' = \frac{q}{Cd}\frac{\mu E}{p}\int_{t_0}^{t}\mathrm{d}t' \implies \Delta U(t) = \frac{q}{C}\frac{\mu}{pd}Et$$

Of course, the above expression holds only for times before the originally-freed ions reach the capacitor plate (i.e. $\Delta U(t)$ does not increase linearly with time indefinitely). Once all ions reached the detector plates, the voltage drop on the capacitor is simply

$$\Delta U = \frac{Q}{C},$$

where Q is the total charge of all ions reaching the detector plates.

4.5 Voltage Drop in a Rectangular Ioniziation Cell

Compute the voltage drop ΔU resulting from the transition of a muon with momentum $pc = 1\,\mathrm{GeV}$ through an parallel-plate ionizing cell with cross-sectional area $S = 1\,\mathrm{dm}^2$ and width $d = 1\,\mathrm{cm}$ filled with argon. Assume the muon travels perpendicularly to the cell's faces. Argon has an ionization energy $w_{\mathrm{i}} = 33.7\,\mathrm{eV}$.

We aim to to compute the accumulated charge on the ionizing cell's faces.

Our first step is to compute the ionizing losses of the muon in the argon gas using the Bethe-Bloch equation, which for review reads

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = K\cdot\frac{\rho Z_{\mathrm{m}}}{A}\cdot\frac{Z_{\mathrm{p}}^2}{\beta^2}\left[\ln\left(\frac{2m_{\mathrm{e}}c^2\beta^2\gamma^2}{Z_{\mathrm{m}}I_0}\right) - \beta^2\right],$$

where $K = 0.3\,\mathrm{MeV\,g^{-1}\,cm^2}$ and $I_0 \sim 10\,\mathrm{eV}$. We simply quote the result:

$$-\frac{\mathrm{d}E}{\mathrm{d}x} \approx 6\,\mathrm{keV\,cm^{-1}}.$$

We find the number of ion pairs N created from the muon's path through the argon gas by dividing the total deposited energy E_{dep} by argon's ionization energy w_{i}:

$$N = \frac{E_{\mathrm{dep}}}{w_{\mathrm{i}}} = \frac{-\frac{\mathrm{d}E}{\mathrm{d}x} \cdot d}{w_{\mathrm{i}}} = \frac{6\,\mathrm{keV\,cm^{-1}} \cdot 1\,\mathrm{cm}}{33.7\,\mathrm{eV}} \approx 180.$$

Assuming each ion carries a charge $q = \pm e_0$, the total charge from these ion pairs is

$$q = N \cdot e_0 = 180 \cdot 1.6 \cdot 10^{-19}\,\mathrm{C} = 2.9 \cdot 10^{-17}\,\mathrm{C}.$$

Next, we compute the cell's capacitance, using the general formula for the capacitance of a parallel-plate capacitor:

$$C = \frac{\epsilon_0 S}{d} = 8.85 \cdot 10^{-12}\,\mathrm{F}.$$

The corresponding voltage drop on the capacitor due to the created ions' charge is

$$\Delta U = \frac{q}{C} = \frac{2.9 \cdot 10^{-17}\,\mathrm{C}}{8.85 \cdot 10^{-12}\,\mathrm{F}} \approx 0.3 \cdot 10^{-5}\,\mathrm{V}.$$

Note how ridiculously small this voltage drop is relative to the roughly 100 V potential difference needed for a 1 cm wide cell.

5 Fifth Exercise Set

5.1 Theory: Multiplication Factor in a Parallel-Plate Cell

An ionization chamber's multiplication factor is the ratio of the (average) number of secondary ions N produced by an incident particle that frees N_0 primary ions. In equation form:

$$M = \frac{N}{N_0}$$

To find M, we need an expression for the number of secondary ionizations N. We will work with electrons instead of positive ions. We begin by letting λ denote the mean free path of electrons in the chamber, but only for collisions which cause ionization. The probability for ionization per unit distance traveled through the chamber is then

$$\alpha = \frac{1}{\lambda}.$$

If N electrons are initially in the chamber, then as these electrons move a distance $\mathrm{d}x$ through the chamber, the number of electrons will increase by

$$\mathrm{d}N = N\alpha \,\mathrm{d}x.$$

This differential equation has the exponential solution

$$N = N_0 e^{\alpha x},$$

where N_0 is the initial number of electrons in the chamber and x is the distance the primary electrons travel through the chamber.

Using this expression for N_0, the corresponding multiplication factor is

$$M = \frac{N}{N_0} = e^{\alpha x}; \qquad M(d) = e^{\alpha d}. \tag{5.1}$$

In other words, we now have a relationship between the multiplication factor M, the distance d, and the ionization parameter α.

We find the factor α, the probability for ionization per unit length, using the *Townsend discharge model*

$$\alpha = pAe^{-\frac{Bp}{E}}, \tag{5.2}$$

where A and B are detector-dependent constants and p and E are pressure and electric field in the chamber, respectively.

5.2 Voltage for a Given Multiplication Factor in a Parallel-Plate Cell

Estimate the voltage U_0 that should be applied to the plates of a parallel-plate ionization cell to achieve a multiplication factor of $M = 10$ over a distance of $d = 1\,\mathrm{cm}$ traveled through the cell from the initial position of ionization. Assume the gas in the ionization cell is at atmospheric pressure. The detector's Townsend discharge model constants are

$$A = 27\,\mathrm{cm}^{-1}\,\mathrm{torr}^{-1} \qquad \text{and} \qquad B = 400\,\mathrm{V}\,\mathrm{cm}^{-1}\,\mathrm{torr}^{-1}.$$

For review, $1\,\text{torr} = 1\,\text{mmHg} \approx 133\,\text{Pa}$.

The electric field for the ionization cell is just the electric field of a parallel-plate capacitor

$$E = \frac{U_0}{d} \implies \alpha = ApE^{-\frac{Bp}{E}} = pAe^{-\frac{dpB}{U_0}}.$$

We find the corresponding multiplication factor from

$$M(d) = e^{\alpha d} \implies \ln M = \alpha d = pAe^{-\frac{Bpd}{U_0}} \cdot d$$

We find the voltage U_0 corresponding to a given M by solving for U_0, which reads

$$\ln\left(\frac{\ln M}{pAd}\right) = -\frac{Bpd}{U_0} \implies U_0 = -\frac{Bpd}{\ln\left(\frac{\ln M}{pAd}\right)}.$$

For the concrete case $M = 10$, the required voltage to achieve $M = 10$ for a parallel-plate cell comes out to

$$U_0 = -\frac{Bpd}{\ln\left(\frac{\ln M}{pAd}\right)} \approx 30\,\text{kV}. \tag{5.3}$$

5.3 Theory: Cylindrical Ionization Chamber

A cylindrical ionization cell consists of cylindrical tube of radius R filled with gas (generally a mixture of noble and organic gas), through the center of which runs a thin wire of radius r_0. The outer cylinder surface is grounded at $U = 0\,\text{V}$ and serves as a cathode, while the inner wire is at positive potential $U = U_0$ and serves as an anode.

From the detector's cylindrical geometry, the electric field in the chamber is

$$E(r) = \frac{U_0}{\ln(R/r_0)}\frac{1}{r},$$

where $r = 0$ is the center of the cylinder. The corresponding potential is

$$\phi(r) = -\frac{U_0}{\ln(R/r_0)} \ln\frac{r}{r_0}.$$

For review from electrostatics, the chamber's capacitance, modeling the chamber as a cylindrical capacitor of length L, is

$$C = \frac{2\pi\epsilon_0 L}{\ln(R/r_0)}.$$

We assume ions move through the gas-filled chamber with constant drift velocity

$$v_{\text{d}}^{(\pm)} = \mu_\pm E,$$

where μ_+ and μ_- are the mobilities of holes and electrons in the gas, respectively.

For orientation: an initial ionizing particle flies into detector and causes primary ionization. The primary ion pairs drift in the electric field; electrons towards the inner anode wire and positive ions toward the outer cathode surface. When primary electrons approach the anode wire (as small r, where the $E \sim 1/r$ electric field is very strong), they cause secondary ionizations which result in a measurable electrical signal indicating the presence of the initial ionizing particle.

5.4 Voltage for a Given Multiplication Factor in a Cylindrical Chamber

Compute the potential difference U required for a multiplication factor $M = 10$ in a cylindrical ionization chamber with outer radius $R = 1\,\mathrm{cm}$ and inner radius $r_0 = 20\,\mu\mathrm{m}$.

In cylindrical geometry the ionization coefficient is a function of electric field and thus position, i.e. $\alpha = \alpha(E) = \alpha(E(r))$. The multiplication factor $M = e^{\alpha x}$ from Equation 5.1 thus generalizes to

$$M = e^{\alpha x} \to M = \exp\left(\int_{r_1}^{r_2} \alpha(r)\,\mathrm{d}r\right), \tag{5.4}$$

where r_1 and r_2 are the radial distance over which the ionizing electrons travel through the chamber.

As before write (absolute value added later with comment because α is probability and should be positive. Supposedly we took that implicitly last time, didn't write aboslute values, and switched the limits of r_0 and r^* integration to account for a minus sign.)

$$\ln M = \left|\int \alpha(r)\,\mathrm{d}r\right|$$

We assume primary ionization occurs at some arbitrary position $r = r^*$, generally of the order $r^* \lesssim R$. The secondary ionization occurs at $r = r_0 + \delta r \approx r_0$, i.e. very near the anode wire. Electrons travel from the point of primary ionization radially inward towards the anode, where they cause secondary ionization.

Using the Townsend discharge model in Equation 5.2, the ionization coefficient α is

$$\alpha(r) = pAe^{-\frac{Bp}{E(r)}}$$

We then take the logarithm of Equation 5.4 and get (note that we switch integration limits so that $\ln M$ comes out to be positive)

$$\begin{aligned}
\ln M = \int_{r^*}^{r_0} \alpha(r)\,\mathrm{d}r \to \int_{r_0}^{r^*} \alpha(r)\,\mathrm{d}r &= Ap\int_{r_0}^{r^*} e^{-\frac{Bp}{E(r)}}\,\mathrm{d}r \\
&= Ap\int_{r_0}^{r^*} \exp\left[-\frac{Bp}{U_0}\ln\frac{R}{r_0}\cdot r\right]\mathrm{d}r \\
&= -\frac{Ap\cdot U_0}{Bp\ln(R/r_0)}\exp\left[-\frac{Bp}{U_0}\ln\frac{R}{r_0}\cdot r\right]_{r_0}^{r^*}.
\end{aligned}$$

We then define the shorthand constant

$$\kappa \equiv \frac{Bp}{U_0}\ln\frac{R}{r_0},$$

in terms of which the multiplication factor expression simplifies to

$$\ln M = -\frac{Ap}{\kappa}\left(e^{-\kappa r^*} - e^{-\kappa r_0}\right).$$

Because $r^* \gg r_0$, the term $e^{-\kappa r^*}$ will fall much more rapidly than $e^{-\kappa r_0}$ for any given (positive) value of κ (the larger r^* causes the exponent to decrease faster). We thus neglect the r_* term, producing

$$\ln M \approx +\frac{Ap}{\kappa} e^{-\kappa r_0} = \frac{ApU_0}{Bp \ln(R/r_0)} \exp\left[-\frac{Bpr_0}{U_0} \ln \frac{R}{r_0}\right].$$

To find the voltage U_0 for a given value of M, we would simply substitute in M and solve the above equation for U_0. Unfortunately, this equation, which takes the general form $0 = a + bxe^{-\frac{c}{x}}$, is not analytically solvable. We would thus solve for U_0 numerically. For our problem's parameters and $M = 10$, the result comes out to

$$U_0 \approx 1.75\,\mathrm{kV}.$$

Lesson: a cylindrical detector requires a much smaller voltage than the 30 kV needed for a multiplication factor $M = 10$ in a parallel-plate cell (see Equation 5.3).

5.5 Time Delay Between Ionization Events in a Cylindrical Chamber

Compute the time delay between the detection of two events in a cylindrical ionization chamber, the first occuring at $r_1 = 0.1R$ and the second at $r_2 = 0.9R$. We are given the chamber's outer and inner radius, which are $R = 1\,\mathrm{cm}$ and $r_0 = 100\,\mathrm{\mu m}$ respectively, the potential difference between the anode and cathode $U_0 = 100\,\mathrm{V}$ and electron mobility $\mu_e = 0.005\,\mathrm{m^2\,V^{-1}\,s^{-1}}$

A detectable ionizing event involves a two-step process:

1. First, an initial ionizing particle flies into a detector and creates a *small* number of primary ions (in this problem, the initial (also called *primary*) ionizations occur at $r = r_1$ and $r = r_2$).
2. Second, electrons produced in the primary ionization drift toward the detector's anode wire under the influence of the detector's electric field; upon closely approaching the anode, where the cylindrical detectors $E \sim 1/r$ electric field is strong, the primary electrons cause secondary ionizations, which creates a *large* number of ions—enough to be detectable. Only the *secondary* ions are numerous enough to create a measurable signal.

With this in mind, the present problem problem reduces to finding the time needed for primary electrons to reach the anode wire at $r \approx r_0$ from the intial position $r = r_{1,2}$; once the primary ions reach the anode, the resulting secondary ions will produce a measurable signal.

We model the electrons as moving through the detector (radially inward, towards the anode wire) with constant drift velocity

$$v_d = -\frac{\mathrm{d}r}{\mathrm{d}t} = \mu_e E = \mu_e \cdot \frac{U_0}{\ln(R/r_0)} \frac{1}{r},$$

where μ_e is electron mobility. We then separate variables and integrate the above expression to get

$$-\int_{r^*}^{r_0} r\,\mathrm{d}r = \frac{\mu_e U_0}{\ln(R/r_0)} \int_0^{t_{\mathrm{sig}}} \mathrm{d}t,$$

where $r^* \in \{r_1, r_2\}$ denotes the position of primary ionization for the first and second particles, while t_{sig} is the to-be-determined time between primary ionization and signal detection. Evaluating the integrals and solving for t_{sig} produces

$$\begin{aligned} t_{\text{sig}} &= \frac{\ln(R/r_0)}{2\mu_e U_0}\left[(r^*)^2 - r_0^2\right] = \frac{\ln(R/r_0)R^2}{2\mu_e U_0}\left[\left(\frac{r^*}{R}\right)^2 - \left(\frac{r_0}{R}\right)^2\right] \\ &\equiv t_0\left[\left(\frac{r^*}{R}\right)^2 - \left(\frac{r_0}{R}\right)^2\right], \end{aligned} \tag{5.5}$$

where, for shorthand, we have packed the constant terms into the coefficient

$$t_0 \equiv \frac{\ln(R/r_0)R^2}{2\mu_e U_0} \approx 46\,\mu\text{s}.$$

Finally, since $r_0 \ll R$, we can neglect the r_0/R contribution to t_{sig} and get

$$t_{\text{sig}} \approx t_0\left(\frac{r^*}{R}\right)^2. \tag{5.6}$$

In our concrete case, where $r^* \in \{r_1 = 0.1R, r_2 = 0.9R\}$, the signal times are

$$t_{\text{sig}}(r_1) = t_0\left(\frac{0.1R}{R}\right)^2 = t_0 \cdot (0.1)^2 \approx 0.46\,\mu\text{s} \tag{5.7}$$

$$t_{\text{sig}}(r_2) = t_0\left(\frac{0.9R}{R}\right)^2 = t_0 \cdot (0.9)^2 \approx 36\,\mu\text{s}. \tag{5.8}$$

Note that the last result for $r^* = 0.9R \sim R$ imposes a minimum time resolution for the detector—this resolution must be of the order $\sim 40\,\mu\text{s}$, since the uncertainty in the position of the primary ionization is of the order $r^* \sim R$. (Since we don't know a priori where in the chamber the initial ionization occurs, we assume an upper bound scenario $r^* \sim R$, which leads to $t_{\text{sig}} \sim 40\,\mu\text{s}$).

6 Sixth Exercise Set

6.1 Signal Dynamics in a Cylindrical Ionization Chamber

Data: We consider a cylindrical ioniziation chamber with outer radius $R = 2\,\mathrm{cm}$*, anode radius,* $r_0 = 20\,\mu\mathrm{m}$*, and length* $L = 20\,\mathrm{cm}$*; the potential difference between cathode and anode is* $U_0 = 0.5\,\mathrm{kV}$*. The electron and ion mobilities in the detector are* $\mu_\mathrm{e} = 400\,\mathrm{cm}^2\,\mathrm{V}^{-1}\,\mathrm{s}^{-1}$ *and* $\mu_\mathrm{i} = 1.5\,\mathrm{cm}^2\,\mathrm{V}^{-1}\,\mathrm{s}^{-1}$*, the average energy required for formation of an electron-ion pair is* $w_\mathrm{i} = 30\,\mathrm{eV}$*, and the chamber's multiplication factor* $M = 10^4$*.*

Problem: Two ionizing particles simultaneously enter the chamber at radii $r_1 = 0.1R$ *and* $r_2 = 0.9R$*. (i) Determine the resulting voltage signal* $U(t)$ *and compute the value of* $U(t)$ *when* $t = 100\,\mu\mathrm{s}$*. Assume both particles deposit energy* $E_1 = E_2 \equiv E_\mathrm{dep} = 100\,\mathrm{keV}$*.*

This problem is a logical continuation of previous problem involving time delay between detected ionizing particles. To review the general situation: an initial ionizing particles frees a small number of *primary* electrons and positive ions in the detector chamber; the primary electrons are accelerated across the chamber's electric potential gradient towards the anode wire, where they gain enough energy to free a large number of *secondary* electron-hole pairs. These secondary ion pairs produce a measurable signal.

We begin by finding time t_sig required by primary electrons to reach the anode from point of primary ionization (and thus the time between primary ionization and initial signal detection). From Equation 5.6 in the previous exercise, this time is

$$t_\mathrm{sig} \approx \frac{\ln(R/r_0)R^2}{2\mu_\mathrm{e}U_0} \cdot \left(\frac{r^*}{R}\right)^2 \equiv t_0 \cdot \left(\frac{r^*}{R}\right)^2,$$

where for the current problem the constant t_0 comes out to

$$t_0 \equiv \frac{\ln(R/r_0)R^2}{2\mu_\mathrm{e}U_0} = \frac{\ln\left[2\,\mathrm{cm}/20\,\mu\mathrm{m}\right] \cdot 4\,\mathrm{cm}^2}{2 \cdot (400\,\mathrm{cm}^2\,\mathrm{V}^{-1}\,\mathrm{s}^{-1}) \cdot (0.5\,\mathrm{kV})} \approx 70\,\mu\mathrm{s}.$$

The signal times for each of the two ionization events is

$$t_\mathrm{sig}(r_1) = t_0 \left(\frac{0.1R}{R}\right)^2 = 70\,\mu\mathrm{s} \cdot (0.1)^2 \approx 0.7\,\mu\mathrm{s}$$
$$t_\mathrm{sig}(r_2) = t_0 \left(\frac{0.9R}{R}\right)^2 = 70\,\mu\mathrm{s} \cdot (0.9)^2 \approx 57\,\mu\mathrm{s}.$$

We now find signal's time evolution. The electric field magnitude E, and corresponding electric potential ϕ, in cylindrical chamber are

$$E(r) = \frac{U_0}{\ln(R/r_0)}\frac{1}{r} \qquad \text{and} \qquad \phi(r) = -\int E(r)\,\mathrm{d}r = -\frac{U_0}{\ln(R/r_0)}\ln\frac{r}{r_0},$$

where we have chosen the potential so that $\phi(r_0) = 0$ at the anode wire. The electric potential energy of a point charge q in the above potential is

$$W_\mathrm{E} = -q\phi(r),$$

and the work $\mathrm{d}W$ done by the electric field in moving a point charge q through a radial displacement $\mathrm{d}r$ is

$$\mathrm{d}W = -\,\mathrm{d}W_{\mathrm{E}} = +q\frac{\mathrm{d}\phi}{\mathrm{d}r}\,\mathrm{d}r = -\frac{qU_0}{\ln(R/r_0)}\left(\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{r}{r_0}\right)\mathrm{d}r = -\frac{qU_0}{\ln(R/r_0)}\frac{\mathrm{d}r}{r};$$

note the intentional use of $\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{r}{r_0} = \frac{\mathrm{d}}{\mathrm{d}r}(\ln r - \ln r_0) = \frac{1}{r}$ instead of $\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{r}{r_0} = \frac{r}{r_0}$ so that $\mathrm{d}W$ has the correct units of energy.

Meanwhile, the ionization chamber acts as a capacitor, with electric field energy given by the general capacitor formula

$$W_{\mathrm{C}} = \frac{1}{2}CU_0^2 \implies \mathrm{d}W_{\mathrm{C}} = CU_0\,\mathrm{d}U = \frac{2\pi\epsilon_0 U_0 L}{\ln(R/r_0)}\,\mathrm{d}U,$$

where we have used the general cylindrical capacitor formula

$$C = \frac{2\pi\epsilon_0 L}{\ln(R/r_0)}. \tag{6.1}$$

By conservation of energy, the work done by the electric force on a charged particle moving through the capacitor's electric field must equal the change in capacitor energy, so we equate $\mathrm{d}W$ to $\mathrm{d}W_{\mathrm{C}}$ to get

$$\mathrm{d}W = \mathrm{d}W_{\mathrm{C}} \implies -\frac{qU_0}{\ln(R/r_0)}\frac{\mathrm{d}r}{r} = \frac{2\pi\epsilon_0 U_0 L}{\ln(R/r_0)}\,\mathrm{d}U.$$

We then solve for the voltage differential $\mathrm{d}U$ to get

$$\mathrm{d}U = -\frac{q}{2\pi\epsilon_0 L}\frac{\mathrm{d}r}{r}. \tag{6.2}$$

Plan: find the voltage $U(t)$ measured by the ionization chamber by integrating the expression for $\mathrm{d}U$ to get $U(r)$ for both secondary electrons and secondary ions, then finding an expressions for $r(t)$ to get $U\big(r(t)\big)$.

We first consider (negatively charged) secondary electrons, which move from the secondary ionization point at $r = r_{\mathrm{s}}^*$ towards the anode wire at $r \approx r_0$. Integrating Equation 6.2 from r_{s}^* to r_0 (and adding an additional negative sign to account for negative electron charge) produces the voltage drop

$$U_{\mathrm{e}} = -\int \mathrm{d}U = +\frac{q}{2\pi\epsilon_0 L}\int_{r_{\mathrm{s}}^*}^{r_0}\frac{\mathrm{d}r}{r} = \frac{q}{2\pi\epsilon_0 L}\ln\frac{r_0}{r_{\mathrm{s}}^*} = -\frac{q}{2\pi\epsilon_0 L}\ln\frac{r_{\mathrm{s}}^*}{r_0}.$$

Meanwhile, the ionization cell voltage drop due to positively-charged ions, which move from the ionization point r_{s}^* to the cylinrder surface at $r = R$, is

$$U_{\mathrm{i}} = \int \mathrm{d}U = -\frac{q}{2\pi\epsilon_0 L}\int_{r_{\mathrm{s}}^*}^{R}\frac{\mathrm{d}r}{r} = -\frac{q}{2\pi\epsilon_0 L}\ln\frac{R}{r_{\mathrm{s}}^*}.$$

Secondary ionziations occur in the neighborhood of the anode wire, so $r_{\mathrm{s}}^* \sim r_0$. Resultantly, $\ln\frac{r_{\mathrm{s}}^*}{r_0} \sim \ln 1 = 0$, meaning the U_{e} contribution of secondary electrons to the ionization chamber's measured signal is negligible; nearly the entire contribution comes from positive ions. For orientation, if $r_0 = 10\,\mu\text{m}$, $R = 1\,\text{cm}$ and $r_\text{s}^* = 11\,\mu\text{m}$ we have $U_\text{e}/U_\text{i} < 0.01$.

Lesson: the total voltage drop U_tot (i.e. the voltage drop measured by a cylindrical detection chamber in response to an incident ionizing particle, which in turn produces primary electrons and ions, of which the primary electrons produce secondary electrons and ions near the anode wire) is well approximated by only the secondary ion-induced voltage drop U_i, i.e.

$$U_\text{tot} = U_\text{e} + U_\text{i} \approx U_\text{i} = -\frac{q}{2\pi\epsilon_0 L}\ln\frac{R}{r_\text{s}^*} \approx -\frac{q}{2\pi\epsilon_0 L}\ln\frac{R}{r_0},$$

where we have made the approximation $r_\text{s}^* \approx r_0$ in the last equality. The corresponding time-dependent signal (replacing R with the time-dependent ion position $r_\text{i}(t)$) is similarly well-approximated by

$$U_\text{tot}(t) = U_\text{e}(t) + U_\text{i}(t) \approx U_\text{i}(t) = -\frac{q}{2\pi\epsilon_0 L}\ln\frac{r_\text{i}(t)}{r_0}.$$

We used U_tot above for clarity; from here forward we will denote the total signal measured by the chamber using just U. To the time-dependent signal $U(t)$, we just need to compute the time-dependent path $r_\text{i}(t)$ of secondary ions through the chamber.

Next: Finding Secondary Ion Position $r_\text{i}(t)$

We approximate $r_\text{i}(t)$ with the ion drift velocity v_i, which we can express using the known electric field and ion mobility.

$$\frac{\mathrm{d}r_\text{i}}{\mathrm{d}t} \approx v_\text{i} = \mu_\text{i} E(r_\text{i}) = \mu_\text{i}\cdot\frac{U_0}{\ln(R/r_0)}\frac{1}{r_\text{i}} = \frac{\mu_\text{i} C U_0}{2\pi\epsilon_0 L}\frac{1}{r_\text{i}} \implies r_\text{i}\,\mathrm{d}r_\text{i} = \frac{\mu_\text{i} C U_0}{2\pi\epsilon_0 L}\,\mathrm{d}t, \tag{6.3}$$

where we have expressed $\ln(R/r_0)$ in terms of the cylindrical chamber's capacitance using Equation 6.1.

Lower integration limit: $r_\text{i}(t_\text{sig}) = r_\text{s}^* \approx r_0$ (secondary ions are created near the anode wire at the secondary ionization time t_sig)

Upper integration limit: $r_\text{i}(t > t_\text{sig}) = r_\text{i}(t)$.

With limits decided, we then integrate Equation 6.3 and get

$$\int_{r_0}^{r_\text{i}} \rho_\text{i}\,\mathrm{d}\rho_\text{i} = \frac{\mu_\text{i} C U_0}{2\pi\epsilon_0 L}\int_{t_\text{sig}}^{t}\mathrm{d}\tau \implies r_\text{i}(t) = \sqrt{r_0^2 + \frac{\mu_\text{i} C U_0}{\pi\epsilon_0 L}(t - t_\text{sig})}.$$

Finally, using the secondary ion trajectory $r_\text{i}(t)$, we solve for the signal $U(t)$:

$$U(t) = -\frac{q_\text{i}}{2\pi\epsilon_0 L}\ln\frac{r_\text{i}(t)}{r_0} = -\frac{q_\text{i}}{2\pi\epsilon_0 L}\ln\left(\frac{1}{r_0}\sqrt{r_0^2 + \frac{\mu_\text{i} C U_0}{\pi\epsilon_0 L}(t - t_\text{sig})}\right),$$

where by using q_i we have explicitly noted that we consider only the charge contribution of positive secondary ions (and not secondary electrons, which contribute negligible signal). We then factor out a coefficient 2 and use the identity $a \ln x = \ln x^a$ to eliminate the square root in the logarithm:

$$U(t) = -\frac{q_\mathrm{i}}{4\pi\epsilon_0 L} \ln\left[1 + \frac{\mu_\mathrm{i} C U_0}{\pi\epsilon_0 L r_0^2} \cdot (t - t_\mathrm{sig})\right] \equiv -\frac{q_\mathrm{i}}{4\pi\epsilon_0 L} \ln\left(1 + \frac{t - t_\mathrm{sig}}{t_0}\right), \tag{6.4}$$

where we have defined the constant t_0, which for our problem comes out to

$$\begin{aligned} t_0 &\equiv \frac{\pi\epsilon_0 L r_0^2}{\mu_\mathrm{i} C U_0} = \frac{\pi\epsilon_0 L r_0^2}{\mu_\mathrm{i} U_0} \cdot \frac{\ln(R/r_0)}{2\pi\epsilon_0 L} = \frac{r_0^2 \ln(R/r_0)}{2\mu_\mathrm{i} U_0} \\ &= \frac{(20\,\mathrm{\mu m})^2 \ln\left[2\,\mathrm{cm}/(20\,\mathrm{\mu m})\right]}{2 \cdot (1.5\,\mathrm{cm^2\,V^{-1}\,s^{-1}}) \cdot (0.5\,\mathrm{kV})} \approx 18\,\mathrm{ns}. \end{aligned} \tag{6.5}$$

Finding Secondary Ion Charge

We now aim to find q, which represents the total secondary charge collected on the ionization cell's surfaces from a single initial ionizing particle.

Plan: find number of primary electron-ion pairs N_p created by initial ionizing particle, then find number of secondary ion pairs N_s created by primary ions.

The number of primary ion pairs equals the energy E_dep deposited by the initial ionizing particle in the chamber divided by the energy w_i need to create a single ion:

$$N_\mathrm{p} = \frac{E_\mathrm{dep}}{W_\mathrm{i}} = \frac{100\,\mathrm{keV}}{30\,\mathrm{eV}} \sim 3 \cdot 10^3$$

We then find the number of ion pairs from the ionization chamber's known multiplication factor M:

$$N_\mathrm{s} = M N_\mathrm{p} = 10^4 \cdot (3 \cdot 10^3) = 3 \cdot 10^7.$$

Since each ion pair contributes a negative electron and positive ion for a total per-pair charge $q_0 = 2e_0$, the total charge from all secondary ion pairs is then

$$q_\mathrm{tot} = N_\mathrm{s} \cdot q_0 = 2 N_\mathrm{s} e_0.$$

The charge contribution of positive ions is half of this, i.e. $q_\mathrm{i} = q_\mathrm{tot}/2 = N_\mathrm{s} e_0$.

With q_i known, the voltage signal in Equation 6.4 is then

$$U(t) = -\frac{N_\mathrm{s} e_0}{4\pi\epsilon_0 L} \begin{cases} \approx 0 & t < t_\mathrm{sig} \\ \ln\left(1 + \frac{t - t_\mathrm{sig}}{t_0}\right) & t_\mathrm{sig} < t < t_\mathrm{sig} + t_\mathrm{ion}. \end{cases} \tag{6.6}$$

Note the generalizations we left out of Equation 6.4:

(i) the signal is negligible before secondary ionizations at $t = t_\mathrm{sig}$ and

(ii) the voltage $U(t)$ does not increase indefinitely, but saturates when all positive ions have reached the chamber's outer cylinder at $t = t_\mathrm{ion}$.

Finally, we compute numerical values. From Equation 6.5, the constant t_0 comes out to $t_0 = 18\,\mathrm{ns}$.

In the context of Equation 5.5, we found the time required by primary electrons to drift from the point of initial ionization r^* to the final anode position r_0 is

$$t_{\mathrm{sig}} = \frac{\ln(R/r_0)}{2\mu_{\mathrm{e}}U_0}\left[(r^*)^2 - r_0^2\right] = \frac{\ln(R/r_0)R^2}{2\mu_{\mathrm{e}}U_0}\left[\left(\frac{r^*}{R}\right)^2 - \left(\frac{r_0}{R}\right)^2\right].$$

Adapted secondary positive ions, which drift from the secondary ionization point $r_{\mathrm{s}}^* \approx r_0$ to the chamber's outer cylinder at $r = R$, this relationship reads

$$t_{\mathrm{ion}} = -\frac{\ln(R/r_0)R^2}{2\mu_{\mathrm{i}}U_0}\left[\left(\frac{r_{\mathrm{s}}^*}{R}\right)^2 - \left(\frac{R}{R}\right)^2\right] \approx +\frac{\ln(R/r_0)R^2}{2\mu_{\mathrm{i}}U_0},$$

where we have made the approximation $r_{\mathrm{s}}^*/R_0 \sim r_0/R_0 \approx 0$. Substituting in numerical values produces

$$t_{\mathrm{ion}} = \ln\left(\frac{2\,\mathrm{cm}}{20\,\mathrm{\mu m}}\right)\frac{(2\,\mathrm{cm})^2}{2\cdot(1.5\,\mathrm{cm^2\,V^{-1}\,s^{-1}})\cdot(0.5\,\mathrm{kV})} \approx 18.4\,\mathrm{ms}.$$

We then define the constant

$$U^* \equiv -\frac{N_{\mathrm{s}}e_0}{4\pi\epsilon_0 L} \approx -0.25\,\mathrm{V},$$

in terms of which the voltage signal in Equation 6.6 reads

$$U(t) = U^*\begin{cases}\approx 0 & t < t_{\mathrm{sig}}\\ \ln\left(1 + \frac{t - t_{\mathrm{sig}}}{t_0}\right) & t_{\mathrm{sig}} < t < t_{\mathrm{sig}} + t_{\mathrm{ion}}.\end{cases}$$

From Equations 5.7 and 5.8, the time taken for primary electrons to drift from primary ionization positions $r_1 = 0.1R$ and $r_2 = 0.9R$ to the anode are

$$t_{\mathrm{sig}}(r_1) \approx 0.46\,\mathrm{\mu s} \approx 0\,\mathrm{\mu s} \qquad \text{and} \qquad t_{\mathrm{sig}}(r_2) \approx 36\,\mathrm{\mu s}.$$

For review, the current problem asks for the voltage signal measured by the cylindrical detector $100\,\mathrm{\mu s}$ after two ionizing particles simultaneously enter the detector chamber at radii $r_1 = 0.1R$ and $r_2 = 0.9R$, respectively. The total voltage is the sum of both particles' contributions, i.e.

$$U_{\mathrm{tot}} = U_1(t) + U_2(t).$$

The voltages contributed by each particle, using $t = 100\,\mathrm{\mu s}$, come out to

$$U^{(1)}(t) = U^*\ln\left(1 + \frac{t - t_{\mathrm{sig}}^{(1)}}{t_0}\right) = U^*\ln\left(1 + \frac{100\,\mathrm{\mu s}}{18\,\mathrm{ns}}\right) \approx 2.2\,\mathrm{V}$$
$$U^{(2)}(t) = U^*\ln\left(1 + \frac{t - t_{\mathrm{sig}}^{(1)}}{t_0}\right) = U^*\ln\left(1 + \frac{44\,\mathrm{\mu s}}{18\,\mathrm{ns}}\right) \approx 2\,\mathrm{V},$$

so the total voltage signal on the detector $100\,\mathrm{\mu s}$ after initial incidence is

$$U(t = 100\,\mathrm{\mu s}) = U^{(1)}(t = 100\,\mathrm{\mu s}) + U^{(2)}(t = 100\,\mathrm{\mu s}) \approx 4.2\,\mathrm{V}.$$

6.2 Energy Resolution in Rectangular Ionization Cell

We consider a rectangular ionization cell of width $d = 10\,\mathrm{cm}$ *filled with 75% argon (density* $\rho_{\mathrm{Ar}} = 1.66\,\mathrm{g\,L^{-1}}$*) and 25% isobuthane (density* $\rho_{\mathrm{ib}} = 2.5\,\mathrm{g\,L^{-1}}$*). The average energy needed to create a single ion pair in argon and isobuthane are* $w_{\mathrm{Ar}} = 26\,\mathrm{eV}$ *and* $w_{\mathrm{but}} = 23\,\mathrm{eV}$*, respectively, while the gases' Fano factors are* $F_{\mathrm{Ar}} = 0.2$ *and* $F_{\mathrm{but}} = 0.2$*.*

Find the cell's energy resolution when measuring a MIP (minimum ionizing particle) muon with normalized ionizing energy losses $2\,\mathrm{MeV\,g^{-1}\,cm^2}$*.*

We define a detector's (relative) energy resolution when measuring a single incident particle's energy as

$$\mathcal{R} = \frac{\sigma_{E_{\mathrm{dep}}}}{E_{\mathrm{dep}}},$$

i.e. the ratio of uncertainy/fluctuations $\sigma_{E_{\mathrm{dep}}}$ in the energy measurement to the energy E_{dep} deposited by the particle in the detector.

We consider two separate cases:

1. If the particle passes through the detector without stopping completely, we find the deposited energy from the particle's ionizing energy losses $-\frac{\mathrm{d}E}{\mathrm{d}x}$.
2. If the incident particle comes to a complete halt in a detector, the deposited energy equals the particles incident kinetic energy.

We find the uncertainty in an incident particle's deposited energy $\sigma_{E_{\mathrm{dep}}}$ from the uncertianty σ_N in the number of created ion pairs.

When a particle passes through the detector without stopping (case 1 above), ion pair production in the detector is a random Poisson process. Because the number of created ion pairs obeys Poisson statistics, the square root of N is proportional to the uncertainty σ_N.

For our problem with a two-gas detector, the deposited energy has two contributions. For argon and isobuthane respectively, these are

$$\begin{aligned}\langle E_{\mathrm{Ar}}\rangle &= 0.75 \cdot d \cdot \frac{\mathrm{d}E}{\mathrm{d}x} \cdot \rho_{\mathrm{Ar}} \\ &= 0.75 \cdot (10\,\mathrm{cm}) \cdot (2\,\mathrm{MeV\,g^{-1}\,cm^2}) \cdot (1.66\,\mathrm{g\,L^{-1}}) \\ &\approx 25\,\mathrm{keV}, \\ \langle E_{\mathrm{ib}}\rangle &= 0.25 \cdot d \cdot \frac{\mathrm{d}E}{\mathrm{d}x} \cdot \rho_{\mathrm{but}} \\ &= 0.25 \cdot (10\,\mathrm{cm}) \cdot (2\,\mathrm{MeV\,g^{-1}\,cm^2}) \cdot (2.5\,\mathrm{g\,L^{-1}}) \\ &\approx 12.5\,\mathrm{keV}.\end{aligned}$$

The corresponding numbers of created ion pairs in each material, using the given ionizition energies w_{i}, are

$$N_{\mathrm{Ar}} = \frac{\langle E_{\mathrm{Ar}}\rangle}{w_{\mathrm{Ar}}} \approx 960 \qquad \text{and} \qquad N_{\mathrm{ib}} = \frac{\langle E_{\mathrm{ib}}\rangle}{w_{\mathrm{ib}}} \approx 520,$$

and so the total number of ion pairs is

$$N_{\mathrm{tot}} = N_{\mathrm{Ar}} + N_{\mathrm{but}} = 960 + 520 = 1480.$$

Case One: Particle Passes Through Detector

Assuming an incident particle only deposits a small amount of its energy into the cell, then ion pair production is random and obeys Poisson statistics, which lets us calculate ion pair uncertainty as

$$\sigma_N = \sqrt{N} = \sqrt{1480} \approx 38.$$

The corresponding relative energy resolution is

$$\mathcal{R} = \frac{\sigma_{E_{\text{dep}}}}{E_{\text{dep}}} = \frac{\sigma_N}{N} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} \approx \frac{1}{38} \approx 0.026.$$

Note that resolution improves (i.e. $\mathcal{R}$ decreases) the larger the number of created ion pairs, or, equivalently, the larger the energy deposited by the ionizing particle in the detector.

Case Two: Particle Stops in Detector

If the particle comes to a complete stop in the ionization cell, then, as the particle begins to slow, pair production is no longer a random independent process. In this case we cannot use Poisson statistics to predict $\sigma_N = \sqrt{N}$, and we make a correction of the form

$$\sigma_N = \sqrt{F \cdot N},$$

where F is the material's Fano factor. Typical Fano factors are less than (often much less than) one, and so resolution improves if the particle deposits its entire energy in the detector.

We now compute energy resolution for case two, assuming the ionizing particle deposited its total energy in the detector.

Assume the particle is a MIP, it has $\beta\gamma \approx 3.5$ and $\beta \approx 0.95$ (rough estimates, just for orientation). We then have (assuming $\gamma \approx 3.5$)

$$T = (\gamma - 1)m_\mu c^2 = 2.5 \cdot 106\,\text{MeV} \approx 265\,\text{MeV}$$

The number of ion pairs is enormous. Assuming an average ionization energy $w_\text{i} \approx 25\,\text{eV}$ for the argon-isobuthane mixture, we have

$$N = \frac{T}{W_\text{i}} \approx 10^7 \implies \mathcal{R} = \frac{\sigma_N}{N} = \frac{\sqrt{F \cdot N}}{N} = \frac{\sqrt{F}}{\sqrt{N}} = \frac{\sqrt{0.2}}{\sqrt{10^7}} \approx 1.4 \cdot 10^{-4}.$$

7 Seventh Exercise Set

7.1 Theory: Measuring Momentum

Basic principle of momentum detectors: place particle in a magnetic field transverse to the particle's velocity and measure the particle's momentum from the curvature of its trajectory in the magnetic field.

We will consider momentum detectors using a central drift chamber configuration. For orientation: two particles traveling in opposite directions along a shared longitudinal axis collide at nominal collision point, around which is centered the drift chamber detector. The entire drift chamber is immersed in a strong magnetic field parallel to the beamline axis.

After the initial collision, the collision products fly through the detector. These particles leave measurable current pulses in position-sensitive tracking modules, on the basis of which we reconstruct their trajectories.

The above described momentum detector can only measure the component of momentum transverse to the direction of the magnetic field; we call this the transverse momentum p_T. Reason: the radius of curvature of a charged particle's trajectory depends only on the quantity $\mathbf{v} \times \mathbf{B}$, which accounts for only the component of $\mathbf{v}$ (and thus momentum $\mathbf{p}$) transverse to $\mathbf{B}$.

We find transverse momentum by equating cetripetal acceleration to Lorentz force

$$\frac{mv_\mathrm{T}^2}{R} = qv_\mathrm{T}B \implies p_\mathrm{T} = qBR,$$

where R is the particle trajectory's radius of curvature in the detector.

If charge q is measured in units of elementary charge e_0, magnetic field B in tesla and radius R in meters, then the particle's transverse momentum p_T in GeV is

$$p_\mathrm{T}c \approx (0.3qBR)\,\mathrm{GeV}. \tag{7.1}$$

Uncertainty in Momentum Measurement

Goal: express uncertainty in transverse momentum σ_{p_T} in terms of uncertainty in the position measurements of a particle's trajectory σ_x.

Geometrical situation: consider two points on a circular trajectory of radius R; connect these two points to circle's center to get a sector spanning angle 2θ. Define chord between the two trajectory points; perpendicular distance from chord's center to circle perimeter is s, given by

$$s = R(1 - \cos\theta) \overset{\theta \ll 1}{\approx} \frac{\theta^2}{2} \cdot R$$

Define L as chord length:

$$R \cdot (2\theta) = L \implies \theta = \frac{L}{2R} \implies s \approx \frac{\theta^2}{2}R = \frac{L^2}{8R}.$$

We then substitute in R from the relationship $p_{\mathrm{T}} = qBR$ to get

$$s = \frac{qL^2B}{8p_{\mathrm{T}}} \implies p_{\mathrm{T}} = \frac{qL^2B}{8s}.$$

To measure s, we need at least three (ideally more) points along the particle's path. We first consider three points, draw the above chord between the first and last point, and let x_i denote the perpendicular displacement of the i-th point from ("above") the chord. In this case s is given by

$$s = x_2 - \frac{x_1 + x_3}{2}$$

If each x_i is measured with same uncertainty σ_x, we find the uncertainty σ_s from

$$\sigma_s^2 = \left(\frac{\partial s}{\partial x_1}\right)^2 \sigma_x^2 + \left(\frac{\partial s}{\partial x_2}\right)^2 \sigma_x^2 + \left(\frac{\partial s}{\partial x_3}\right)^2 \sigma_x^2 = \frac{3}{2}\sigma_x^2$$

We find the corresponding relative uncertainty in transverse momentum from

$$\sigma_{p_{\mathrm{T}}} = \left|\frac{\partial p_{\mathrm{T}}}{\partial s}\right| \sigma_s \propto \frac{1}{s^2}\sigma_s \implies \frac{\sigma_{p_{\mathrm{T}}}}{p_{\mathrm{T}}} = \frac{\sigma_s}{s}.$$

Finally, substituting in the expressions for σ_s and s produces

$$\frac{\sigma_{p_{\mathrm{T}}}}{p_{\mathrm{T}}} = \sqrt{\frac{3}{2}}\sigma_x \cdot \frac{8p_{\mathrm{T}}}{qBL^2} = \frac{\sqrt{96}\sigma_x}{qBL^2} \cdot p_{\mathrm{T}}.$$

Lesson: momentum resolution worsens (increases) with increasing transverse momentum (which leads to a larger radius of curvature). Keep in mind the above expression holds only under the assumption $\theta \ll 1$, i.e. that the three trajectory points are close together.

Without derivation, an expression for momentum resolution using N points on a particle's trajectory, each measured with uncertainty σ_x, is

$$\frac{\sigma_{p_{\mathrm{T}}}}{p_{\mathrm{T}}} = \frac{\sigma_x}{qBL^2} \cdot \sqrt{\frac{720}{N+4}} \cdot p_{\mathrm{T}}$$

As might be expected, the momentum resolution improves with increasing N.

7.2 Identifying Mass from Momentum and Energy Measurements

An unknown charged particle leaves a radius of curvature $R = 7.25\,\mathrm{m}$ in a cylindrical detector with a magnetic field $B = 1\,\mathrm{T}$ along the beam axis. Independently, the particle's kinetic energy is measured by a calorimeter as $T = (2.00 \pm 0.03)\,\mathrm{GeV}$. Assuming uncertainty in R is negligible, identify the resulting particle.

We first estimate the particle's momentum using Equation 7.1, which gives

$$p_{\mathrm{T}}c \approx 0.3 \cdot q\,[e_0] \cdot B\,[\mathrm{T}] \cdot R\,[\mathrm{m}] \cdot \mathrm{GeV} = 0.3 \cdot q\,[e_0] \cdot 1 \cdot 7.25 \cdot \mathrm{GeV} \equiv |z| \cdot 2.175\,\mathrm{GeV},$$

where we have used z to denote the particle's charge in units of e_0.

We then find an independent expression for the particle's mass using the relationship

$$m^2c^4 = E^2 - p^2c^2 = (T + mc^2)^2 - p^2c^2 \implies mc^2 = \frac{p^2c^2 - T^2}{2T},$$

Using $p_{\mathrm{T}}c = |z| \cdot 2.175\,\mathrm{GeV}$, the particle's mass is then

$$mc^2 = \frac{p^2c^2 - T^2}{2T} = \frac{z^2 \cdot (2.175\,\mathrm{GeV})^2 - (2.0\,\mathrm{GeV})^2}{2 \cdot 2.0\,\mathrm{GeV}} = \left(\frac{2.175^2}{4}z^2 - 1\right)\mathrm{GeV}.$$

The corresponding uncertainty in the particle's mass is

$$\begin{aligned}\sigma_m c^2 &= \left|\frac{\partial m}{\partial T}\right|\sigma_T = \left|\frac{-2T \cdot 2T - 2(p^2c^2 - T^2)}{4T^2}\right| \cdot \sigma_T = \frac{1}{2}\left(1 + \frac{(pc)^2}{T^2}\right)\sigma_T \\ &= \frac{1}{2}\left(1 + \frac{(2.175\,\mathrm{GeV})^2}{(2\,\mathrm{GeV})^2}z^2\right) \cdot 0.03\,\mathrm{GeV}.\end{aligned}$$

The particle's charge isn't known; testing $z = 1$ and $z = 2$ produces

$$\begin{aligned} m &= (210 \pm 32)\,\mathrm{MeV} \qquad (z = 1), \\ m &= (3.84 \pm 0.09)\,\mathrm{GeV} \qquad (z = 2). \end{aligned}$$

The result for $z = 1$ is not particularly close to the mass of any known particle, but the mass for $z = 2$ is reasonably close to the mass of an alpha particle, which has $m_\alpha \approx 3.73\,\mathrm{GeV}$.

8 Eighth Exercise Set

In the next few exercises, we will review the basics of electron behavior in semiconductors as a theoretical background for understanding semiconducting particle detectors.

8.1 Theory: Important Quantities in Intrinsic Semiconductors

Note: We assume the material in the next few exercises is familiar from Solid State Physics. Thus, we merely summarize and do not explain the origin of the equations in detail.

Some relevant quantities for intrinsic semiconductors:

- Let E_v denote the top of the valence band and E_c the bottom of the conduction band.
- Let $E_\mathrm{g} \equiv E_\mathrm{c} - E_\mathrm{v}$ denote the energy gap between the valence and conduction bands. By convention we set $E_\mathrm{v} \equiv 0\,\mathrm{eV}$ and so $E_\mathrm{c} = E_\mathrm{g}$.
- The chemical potential μ in an intrinsic semiconductor obeys

$$\mu = E_\mathrm{v} + \frac{E_\mathrm{g}}{2} + \frac{3}{4} k_B T \ln \frac{m_\mathrm{v}^*}{m_\mathrm{c}^*},$$

  where m_c^* and m_v^* are the effective masses of conduction band electrons and valence band holes, respectively.

  The Fermi energy in an intrinsic semiconductor occurs at $E_\mathrm{F} = E_\mathrm{g}/2$.

The Fermi-Dirac distribution, which gives the probability that an electron state with energy E is occupied in a free electron gas at temperature T, is

$$f(E) = \frac{1}{e^{\beta(E-\mu)} + 1}, \quad \text{where } \beta \equiv k_B T.$$

For orientation, $k_B T \approx 0.025\,\mathrm{eV}$ at $T = 300\,\mathrm{K}$. In the regime $k_B T \ll E_\mathrm{g}$, which generally applies in everyday conditions, the Fermi-Dirac distribution is approximately

$$f(E) \sim e^{-\beta(E-E_\mathrm{F})} \qquad (\text{if } k_B T \ll E_\mathrm{g}).$$

Without derivation, the densities of states for conduction band electrons and valence band holes are approximately

$$g_\mathrm{c}(E) \approx \frac{1}{2\pi^2}\left(\frac{2m_\mathrm{c}^*}{\hbar^2}\right)^{3/2}\sqrt{|E - E_\mathrm{c}|},$$

$$g_\mathrm{v}(E) \approx \frac{1}{2\pi^2}\left(\frac{2m_\mathrm{v}^*}{\hbar^2}\right)^{3/2}\sqrt{|E - E_\mathrm{v}|}.$$

These expressions assume $E_\mathrm{c} - \mu \ll k_B T$ and $|E_\mathrm{g} - \mu| \ll k_B T$, that the semiconductor's effective mass tensor has equal eigenvalues, and that the conduction and valence band dispersions are approximately quadratic. (All of these conditions are satisfied in silicon in everyday conditions.) For more information see the notes from FMF's Solid State Physics **TODO:** add link

Again without derivation, the number densities (number of charge carriers per unit volume) for conduction band electrons and valence band holes are

$$n_{\mathrm{c}} = \frac{1}{4}\left(\frac{2m_{\mathrm{c}}^{*}k_{B}T}{\pi\hbar^{2}}\right)^{3/2} e^{-\beta(E_{\mathrm{c}}-\mu)} \equiv N_{\mathrm{c}}(T)e^{-\beta(E_{\mathrm{c}}-\mu)}$$
$$p_{\mathrm{v}} = \frac{1}{4}\left(\frac{2m_{\mathrm{v}}^{*}k_{B}T}{\pi\hbar^{2}}\right)^{3/2} e^{-\beta(\mu-E_{\mathrm{v}})} \equiv P_{\mathrm{v}}(T)e^{-\beta(\mu-E_{\mathrm{v}})}.$$

Intrinsic semiconductors obey the equality

$$n_{\mathrm{c}} = p_{\mathrm{v}} \equiv n_{\mathrm{i}},$$

where have introduced n_{i} to denote the number density of charge carriers in an *intrinsic* semiconductor. From the above expressions for n_{c} and p_{v}, we then derive

$$n_{\mathrm{i}}^{2} = N_{\mathrm{c}}P_{\mathrm{v}}e^{-\beta E_{\mathrm{g}}} \implies n_{\mathrm{i}} = \sqrt{N_{\mathrm{c}}P_{\mathrm{v}}}e^{-\frac{\beta E_{\mathrm{g}}}{2}} = \frac{1}{4}\left(\frac{2k_{B}T\sqrt{m_{\mathrm{e}}^{*}m_{\mathrm{h}}^{*}}}{\pi\hbar^{2}}\right)^{3/2} e^{-\frac{\beta E_{\mathrm{g}}}{2}}.$$

Room-temperature values of effective mass m^{*}, electron/hole mobility μ_{e} and μ_{h} (not to be confused with chemical potential μ, which does not have an index) and band gap E_{g} in silicon and germanium are given in Table 1 below.

Quantity	Si	Ge
$m_{\mathrm{c}}^{*}/m_{\mathrm{e}}$	0.26	0.12
$m_{\mathrm{v}}^{*}/m_{\mathrm{e}}$	0.36	0.21
μ_{e} $[\mathrm{m^2\,V^{-1}\,s^{-1}}]$	0.16	0.38
μ_{h} $[\mathrm{m^2\,V^{-1}\,s^{-1}}]$	0.04	0.18
E_{g} [eV]	1.12	0.66

Table 1: Numerical values of some semiconductor-related quantities in silicon and germanium at room temperature $T \sim 300\,\mathrm{K}$.

For later use, the intrinsic carrier density in silicon at room temperature is

$$n_{\mathrm{i}} = \frac{1}{4}\left(\frac{2\cdot 0.025\,\mathrm{eV}\cdot 0.5\,\mathrm{MeV/c^2}\cdot 0.3}{\pi\hbar^{2}}\right)^{3/2} e^{-\frac{1.22\,\mathrm{eV}}{2\cdot 0.025\,\mathrm{eV}}} \sim 10^{16}\,\mathrm{m^{-3}}.$$

8.2 Resistivity of an Intrinsic and Doped Semiconductor

Determine the resistivity of: (i) an intrinsic silicon semiconductor and (ii) an n*-type silicon semiconductor doped with* $N_{\mathrm{d}} = 10^{18}$ *donors.*

8.2.1 Resistivity of an Intrinsic Semiconductor

From Ohm's law, electric current density j in the semiconductor exposed to an external electric field E is

$$j = \sigma_{\mathrm{E}} E,$$

where σ_{E} is the semiconductor's electrical conductivity. Independently, the current density in the conducting material with charge carriers of charge q is

$$j = nqv_{\mathrm{d}},$$

where n and v_{d} are the charge carrier density and drift velocity, respectively.

The drift velocity is in turn modeled by the relationship

$$v_{\mathrm{d}} = \mu E,$$

where μ is a proportionality constant called charge carrier mobility. Equating $j = \sigma_{\mathrm{E}} E$ and $j = nqv_{\mathrm{d}}$ and substituting in $v_{\mathrm{d}} = \mu E$ gives the expression

$$\sigma_{\mathrm{E}} = nq\mu,$$

from which we can find the semiconductor's resistivity ρ_{E} via

$$\rho_{\mathrm{E}} = \frac{1}{\sigma_{\mathrm{E}}} = \frac{1}{nq\mu}.$$

We write the total current in a semiconductor as the sum of both hole and electron contributions:

$$j = e_0 n_{\mathrm{h}} v_{\mathrm{h}} - (-e_0) n_{\mathrm{e}} v_{\mathrm{e}} = e_0 n_{\mathrm{i}} (\mu_{\mathrm{e}} + \mu_{\mathrm{h}}) E,$$

where we have used the intrinsic semiconductor identity $n_{\mathrm{e}} = n_{\mathrm{h}} \equiv n_{\mathrm{i}}$. The intrinsic semiconductor's resistivity is then

$$\sigma_{\mathrm{E}} = e_0 n_{\mathrm{i}} (\mu_{\mathrm{e}} + \mu_{\mathrm{h}}) \quad \Longrightarrow \quad \rho_{\mathrm{E}} = \frac{1}{\sigma_{\mathrm{E}}} = \frac{1}{e_0 n_{\mathrm{i}} (\mu_{\mathrm{e}} + \mu_{\mathrm{h}})}.$$

Substituting in numerical values from Table 1 produces

$$\rho_{\mathrm{E}} = \frac{1}{(1.6 \cdot 10^{-19}\,\mathrm{C}) \cdot (10^{16}\,\mathrm{m}^{-3}) \cdot (0.2\,\mathrm{m}^2\,\mathrm{V}^{-1}\,\mathrm{s}^{-1})} \approx 3 \cdot 10^3\,\Omega\,\mathrm{m}$$

8.2.2 Resistivity of a Doped Semiconductor

The dopant density in this problem is $N_{\mathrm{d}} \sim 10^{18}\,\mathrm{m}^{-3} \gg n_{\mathrm{i}} \sim 10^{16}\,\mathrm{m}^{-3}$.

At a temperature $T = 300\,\mathrm{K}$, nearly all *donor* electrons are in the conducting band. In this case, assuming $N_{\mathrm{d}} \gg n_{\mathrm{i}}$, the number density of all charge carriers is roughly N_{d}—the number of donor electrons in the conducting band.

We thus approximate the doped semiconductor's charge carrier density n_{all} with the dopant density N_{d}, which gives an electric current density

$$j = n_{\mathrm{all}} \cdot e_0 \mu_{\mathrm{e}} E \approx N_{\mathrm{d}} e_0 \mu_{\mathrm{e}} E$$

We then equate $j \approx N_\mathrm{d} e_0 \mu_\mathrm{e} E$ to $j = \sigma_\mathrm{E} E = E/\rho_\mathrm{E}$ to get

$$\rho_\mathrm{E}^{(\mathrm{n})} = \frac{1}{e_0 N_\mathrm{d} \mu_\mathrm{e}} = \frac{1}{(1.6 \cdot 10^{-19}\,\mathrm{C}) \cdot (10^{18}\,\mathrm{m}^{-3}) \cdot (0.16\,\mathrm{m}^2\,\mathrm{V}^{-1}\,\mathrm{s}^{-1})} \approx 30\,\Omega\,\mathrm{m} \tag{8.1}$$

Note the doped semiconductor's resistivity is much lass than the intrinsic semiconductor result $\rho_\mathrm{E}^{(\mathrm{i})} \approx 3 \cdot 10^3\,\Omega\,\mathrm{m}$.

In passing, we note that a p-type semiconductor with the same dopant density $N_\mathrm{a} = 10^{18}\,\mathrm{m}^{-3}$ would have a resistivity

$$\rho_\mathrm{E}^{(\mathrm{p})} = \frac{1}{e_0 N_\mathrm{a} \mu_\mathrm{h}} = \frac{1}{(1.6 \cdot 10^{-19}\,\mathrm{C}) \cdot (10^{18}\,\mathrm{m}^{-3}) \cdot (0.04\,\mathrm{m}^2\,\mathrm{V}^{-1}\,\mathrm{s}^{-1})} \approx 120\,\Omega\,\mathrm{m}. \tag{8.2}$$

8.3 Depletion Region Width in a p-n Junction

Derive the width of the depletion region in semiconducting p-n *junction. Analyze the limit cases* $N_\mathrm{a} \gg N_\mathrm{d}$ *and* $N_\mathrm{d} \gg N_\mathrm{a}$.

For simplicity, we assume the charge density ρ in the depletion region is a step function of the form

$$\rho(x) = \begin{cases} -e_0 N_\mathrm{a} & x \in (-x_\mathrm{p}, 0) \\ e_0 N_\mathrm{d} & x \in (0, x_\mathrm{n}) \end{cases} \tag{8.3}$$

where the depletion region spans the x range $(-x_\mathrm{p}, x_\mathrm{n})$. We begin our analysis with the Poisson equation from electrostatics, which reads

$$\nabla^2 \phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon\epsilon_0},$$

and in a one-dimensional case reduces to

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = -\frac{\rho(x)}{\epsilon\epsilon_0} \implies \frac{\mathrm{d}\phi}{\mathrm{d}x} = -\int \frac{\rho(x)}{\epsilon\epsilon_0}\,\mathrm{d}x$$

We then integrate the charge density in Equation 8.3 from $-x_\mathrm{p}$ to x_n to get

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \begin{cases} \frac{e_0 N_\mathrm{a}}{\epsilon\epsilon_0} x + c_\mathrm{p} & x \in (-x_\mathrm{p}, 0) \\ -\frac{e_0 N_\mathrm{d}}{\epsilon\epsilon_0} x + c_\mathrm{n} & x \in (0, x_\mathrm{n}). \end{cases}$$

As boundary conditions we require $\frac{\mathrm{d}\phi}{\mathrm{d}x} = 0$ at $x = x_\mathrm{n}$ and at $x = -x_\mathrm{p}$, producing

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \begin{cases} \frac{e_0 N_\mathrm{a}}{\epsilon\epsilon_0} (x + x_\mathrm{p}) & x \in (-x_\mathrm{p}, 0) \\ -\frac{e_0 N_\mathrm{d}}{\epsilon\epsilon_0} (x - x_\mathrm{n}) & x \in (0, x_\mathrm{n}). \end{cases} \tag{8.4}$$

We then integrate once more to get the deplection region potential

$$\phi(x) = \begin{cases} \frac{e_0 N_\mathrm{a}}{\epsilon\epsilon_0} \left(\frac{x^2}{2} + x_\mathrm{p} x\right) + c_\mathrm{p} & x \in (-x_\mathrm{p}, 0) \\ -\frac{e_0 N_\mathrm{d}}{\epsilon\epsilon_0} \left(\frac{x^2}{2} - x_\mathrm{n} x\right) + c_\mathrm{n} & x \in (0, x_\mathrm{n}). \end{cases}$$

As a first boundary condition we require continuity $\phi(x)$ at the p-n boundary $x = 0$, which produces $c_\mathrm{p} = c_\mathrm{n} \equiv c$. As a second boundary condition we require $\phi(-x_\mathrm{p}) = 0$

and $\phi(x_\mathrm{n}) - \phi(-x_\mathrm{p}) = V_0$, (where V_0 is the potential difference across the depletion region). The second boundary conditions leads to

$$\phi(x_\mathrm{n}) = V_0 = \frac{e_0 N_\mathrm{d}}{2\epsilon\epsilon_0} x_\mathrm{n}^2 + c \qquad \text{and} \qquad V(-x_\mathrm{p}) = 0 = \frac{-e_0 N_\mathrm{a}}{2\epsilon\epsilon_0} x_\mathrm{p}^2 + c,$$

Which we then solve for the integration constant c to get

$$c = \frac{e_0 N_\mathrm{a}}{2\epsilon\epsilon_0} x_\mathrm{p}^2.$$

Substituting c into $\phi(x_\mathrm{n})$ leads to

$$\phi(x_\mathrm{n}) = V_0 = \frac{e_0 N_\mathrm{d}}{2\epsilon\epsilon_0} x_\mathrm{n}^2 + \frac{e_0 N_\mathrm{a}}{2\epsilon\epsilon_0} x_\mathrm{p}^2 = \frac{e_0}{2\epsilon\epsilon_0} \left(N_\mathrm{d} x_\mathrm{n}^2 + N_\mathrm{a} x_\mathrm{p}^2 \right). \tag{8.5}$$

With $c_\mathrm{n} = c_\mathrm{p} \equiv c$ and V_0 known, the depletion region potential comes out to

$$\phi(x) = \begin{cases} \frac{e_0 N_\mathrm{a}}{2\epsilon\epsilon_0}(x + x_\mathrm{p})^2 & x \in (-x_\mathrm{p}, 0) \\ V_0 - \frac{e_0 N_\mathrm{d}}{2\epsilon\epsilon_0}(x - x_\mathrm{n})^2 & x \in (0, x_\mathrm{n}) \end{cases}$$

Finally, we recall the p-n junction conservation of charge condition

$$N_\mathrm{a} x_\mathrm{p} = N_\mathrm{d} x_\mathrm{n},$$

which we combine Equation 8.5 to solve for x_n and x_p. The result is

$$x_\mathrm{n} = \sqrt{\frac{2\epsilon\epsilon_0 V_0}{e_0 N_\mathrm{d}\left(1 + \frac{N_\mathrm{d}}{N_\mathrm{a}}\right)}} \qquad \text{and} \qquad x_\mathrm{p} = \sqrt{\frac{2\epsilon\epsilon_0 V_0}{e_0 N_\mathrm{a}\left(1 + \frac{N_\mathrm{a}}{N_\mathrm{d}}\right)}}.$$

The total depletion region width d_pn is then

$$d_\mathrm{pn} = x_\mathrm{n} + x_\mathrm{p} = \sqrt{\frac{2\epsilon\epsilon_0 V_0}{e_0} \frac{N_\mathrm{a} + N_\mathrm{d}}{N_\mathrm{a} N_\mathrm{d}}}. \tag{8.6}$$

Note that if $N_\mathrm{a} \gg N_\mathrm{d}$ then $x_\mathrm{n} \gg x_\mathrm{p}$ and thus $d_\mathrm{pn} \approx x_\mathrm{n}$ which is

$$d_\mathrm{pn} \approx x_\mathrm{n} \approx \sqrt{\frac{2\epsilon\epsilon_0 V_0}{e_0 N_\mathrm{d}}} \qquad (\text{if } N_\mathrm{a} \gg N_\mathrm{d}). \tag{8.7}$$

Analogously, if $N_\mathrm{d} \gg N_\mathrm{a}$ then $x_\mathrm{p} \gg x_\mathrm{n}$ and thus $d_\mathrm{pn} \approx x_\mathrm{p}$ which is

$$d_\mathrm{pn} \approx x_\mathrm{p} \approx \sqrt{\frac{2\epsilon\epsilon_0 V_0}{e_0 N_\mathrm{a}}} \qquad (\text{if } N_\mathrm{d} \gg N_\mathrm{a}). \tag{8.8}$$

9 Ninth Exercise Set

9.1 Theory: Relating Resistivity and Depletion Region Width

From Equations 8.1 and 8.2 in the previous exercise set, recall that the electrical resistivities ρ_{E} in an n-type and p-type semiconductor are approximately

$$\rho_{\mathrm{E}}^{(\mathrm{n})} = \frac{1}{e_0 N_{\mathrm{d}} \mu_{\mathrm{e}}} \qquad \text{and} \qquad \rho_{\mathrm{E}}^{(\mathrm{p})} = \frac{1}{e_0 N_{\mathrm{a}} \mu_{\mathrm{h}}}, \tag{9.1}$$

where μ_{e} and μ_{h} denote electron and hole mobilities, respectively. If we invert these expressions and solve for N_{d} or N_{a}, we can estimate p-n junction's width from the Equations 8.7 and 8.7 in the previous exercise set, which for review read

$$d_{\mathrm{pn}} \approx x_{\mathrm{n}} = \sqrt{\frac{2\epsilon\epsilon_0 V_0}{e_0 N_{\mathrm{d}}}} \qquad (\text{if } N_{\mathrm{a}} \gg N_{\mathrm{d}}),$$
$$d_{\mathrm{pn}} \approx x_{\mathrm{p}} = \sqrt{\frac{2\epsilon\epsilon_0 V_0}{e_0 N_{\mathrm{a}}}} \qquad (\text{if } N_{\mathrm{d}} \gg N_{\mathrm{a}}).$$

Solving for N_{d} and N_{a} in Equation 9.1 and substituting in above leads to

$$d_{\mathrm{pn}} \sim \sqrt{2\epsilon\epsilon_0 \rho_{\mathrm{n}} \mu_{\mathrm{e}} V_0} \qquad (\text{if } N_{\mathrm{a}} \gg N_{\mathrm{d}}),$$
$$d_{\mathrm{pn}} \sim \sqrt{2\epsilon\epsilon_0 \rho_{\mathrm{p}} \mu_{\mathrm{h}} V_0} \qquad (\text{if } N_{\mathrm{d}} \gg N_{\mathrm{a}}),$$

where we have written resitivity without an $_{\mathrm{E}}$ subscript for conciseness.

Typical dielectric constants are $\epsilon_{\mathrm{Si}} \approx 12$ in silicon and $\epsilon_{\mathrm{Ge}} \approx 16$ in germanium, from which we can estimate typical p-n junction widths in a silicon semiconductor. Assuming ρ is measured in $\Omega\,\mathrm{cm}$ and V_0 in volts, we have

$$d_{\mathrm{Si}} \approx 0.53\sqrt{\rho_{\mathrm{n}} V_0} \cdot \mu\mathrm{m} \qquad (\text{if } N_{\mathrm{a}} \gg N_{\mathrm{d}}),$$
$$d_{\mathrm{Si}} \approx 0.32\sqrt{\rho_{\mathrm{p}} V_0} \cdot \mu\mathrm{m} \qquad (\text{if } N_{\mathrm{d}} \gg N_{\mathrm{a}}).$$

Corresponding values for germanium-based p-n junctions are

$$d_{\mathrm{Ge}} \approx 1.00\sqrt{\rho_{\mathrm{n}} V_0} \cdot \mu\mathrm{m} \qquad (\text{if } N_{\mathrm{a}} \gg N_{\mathrm{d}}),$$
$$d_{\mathrm{Ge}} \approx 0.65\sqrt{\rho_{\mathrm{p}} V_0} \cdot \mu\mathrm{m} \qquad (\text{if } N_{\mathrm{d}} \gg N_{\mathrm{a}}).$$

For for silicon and germanium, at a depletion region voltage $V_0 \sim 1\,\mathrm{V}$ and resistivity $\rho \sim 100\,\Omega\,\mathrm{cm}$ the deplection region width is of the order $d \sim 50\,\mu\mathrm{m}$. This is small! More specifically, for use in particle detectors we desire a large depletion region, which provides a large region in which an incident particle can deposit energy and free electron-hole pairs leading to better energy and position resolution.

For a Larger Depletion Region

We can increase the width of a depletion region by applying an external *bias* voltage V_{b} in the reverse bias direction. Typically V_{b} is of the order of $100\,\mathrm{V}$ (perhaps up to $300\,\mathrm{V}$) which increases the depletion region width to $d \sim 1$ to $5\,\mathrm{mm}$ for typical resistivities. Values of V_{b} are restricted to about $300\,\mathrm{V}$ because of increasing leakage current (causing electrical noise) and risk of dielectric breakdown at high voltage.

In the presence of reverse baising, the deplection region width increases to

$$d_{\text{pn}} = d_0\sqrt{1 + \frac{V_{\text{b}}}{V_0}},$$

where d_0 is the width without biasing (given in Equation 8.6).

9.2 Energy Resolution in a Semiconducting Detector

We consider a (i) silicon and (ii) germanium semiconducting detector of thickness $d = 1\,\text{mm}$ *with Fano factor* $F = 0.12$*. The energies required for formation of a single electron-hole pair are* $w_{\text{Si}} = 3.63\,\text{eV}$ *in silicon and* $w_{\text{Ge}} = 2.96\,\text{eV}$ *in germanium, while the particle's ionizing losses in the conductor are known to be* $\frac{\mathrm{d}E_{\text{Si}}}{\mathrm{d}x} = 4\,\text{MeV}\,\text{cm}^{-1}$ *in silicon and* $\frac{\mathrm{d}E_{\text{Ge}}}{\mathrm{d}x} = 7.3\,\text{MeV}\,\text{cm}^{-1}$ *in germanium. Determine the detector's relative energy resolution for a MIP muon.*

As in Exercise 6.2, the detector's relative energy resolution when measuring a particle depositing energy E_{dep} in the detector is

$$\mathcal{R} = \frac{\sigma_{E_{\text{dep}}}}{E_{\text{dep}}},$$

where $\sigma_{E_{\text{dep}}}$ is the uncertainty/fluctuations in the measurement of $\sigma_{E_{\text{dep}}}$. As in Exercise 6.2, we consider two cases:

1. If the particle passes through the detector without stopping, electron-hole pair production is a random Poisson process ($\sigma_N = \sqrt{N}$), and we estimate energy resolution from fluctuations in the number of produced electron-hole pairs via

$$\frac{\sigma_{E_{\text{dep}}}}{E_{\text{dep}}} = \frac{\sigma_N}{N} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}.$$

2. If the incident particle comes to a complete halt in a detector, pair production is no longer a random process as the particle slows. In this case we estimate energy resolution with the Fano factor correction $\sigma_N \to \sqrt{FN}$. This gives

$$\frac{\sigma_{E_{\text{dep}}}}{E_{\text{dep}}} = \frac{\sigma_N}{N} = \frac{\sqrt{FN}}{N} = \sqrt{\frac{F}{N}}.$$

To determine which case applies to our problem, we first compute the energy deposited by the muon in the (i) Si or (ii) Ge detector. Using the given ionizing energy losses, the deposited energies are

$$E_{\text{dep}}^{(\text{Si})} = \frac{\mathrm{d}E_{\text{Si}}}{\mathrm{d}x} \cdot d = (4.0\,\text{MeV}\,\text{cm}^{-1}) \cdot (1\,\text{mm}) \approx 0.4\,\text{MeV},$$
$$E_{\text{dep}}^{(\text{Ge})} = \frac{\mathrm{d}E_{\text{Ge}}}{\mathrm{d}x} \cdot d = (7.3\,\text{MeV}\,\text{cm}^{-1}) \cdot (1\,\text{mm}) \approx 0.7\,\text{MeV}.$$

We then compare these values to the muon's incident kinetic energy. Since the muon is known to be a minimum ionizing particle, its speed is $\gamma\beta \sim 3$, from which, using $m_\mu \sim 108\,\text{MeV}$, we can estimate the muon's kinetic energy with

$$\gamma\beta = \frac{pc}{m_\mu c^2} \implies pc \sim 330\,\text{MeV}.$$

The muon's kinetic energy is thus of the order $T \sim 100\,\mathrm{MeV}$ (much larger than deposited energy), so the muon will pass through the detector and loose only a small amount of its initial energy. We can thus assume random, Poisson-distributed ion pair production where $\sigma_N = \sqrt{N}$, in which case the detector's energy resolution is

$$\frac{\sigma_{E_\mathrm{dep}}}{E_\mathrm{dep}} = \frac{\sigma_N}{N} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}.$$

We find the number of created electron-hole pairs from

$$N = \frac{E_\mathrm{dep}}{w_\mathrm{i}} \implies \mathcal{R} = \frac{\sigma_E}{E} = \sqrt{\frac{w_\mathrm{i}}{E_\mathrm{dep}}},$$

where w_i is the energy needed to create a single ion (electron-hole) pair in the detector. Substituting in given numerical values produces

$$\mathcal{R}_\mathrm{Si} = \sqrt{\frac{w_\mathrm{Si}}{E_\mathrm{dep}^{(\mathrm{Si})}}} = \sqrt{\frac{3.63\,\mathrm{eV}}{0.4 \cdot 10^6\,\mathrm{eV}}} \approx 3 \cdot 10^{-3},$$

$$\mathcal{R}_\mathrm{Ge} = \sqrt{\frac{w_\mathrm{Ge}}{E_\mathrm{dep}^{(\mathrm{Ge})}}} = \sqrt{\frac{2.96\,\mathrm{eV}}{0.7 \cdot 10^6\,\mathrm{eV}}} \approx 2 \cdot 10^{-3}.$$

9.3 Signal Dynamics in a p-n Semiconducting Detector

Consider a p-n semiconductor with a heavily-doped n region and a weakly doped p region (so that $N_d \gg N_a$ and $x_p \gg x_n$). We apply a positive voltage V_0 to the n-type electrode and ground the p-type electrode. Derive the time-dependent current and voltage signals $I(t)$ and $U(t)$ registed by detector after an incident ionizing particle frees an electron-hole pair in the semiconductor's depletion region.

9.3.1 Electrical Current

For orientation: an incident ionizing particle deposits energy in the depletion region and produces an electron-hole pair. The depletion region's internal electric field sweeps the electron towards the n-type region and the hole towards the p-type region. The electron and hole produce a measurable current pulse indicating the presence of the initial incident particle.

We use a one dimensional coordinate system along the x axis. In order of increasing x, the detector consists of a p region, depletion region and n region, where the depletion region spans $x \in (-x_\mathrm{p}, x_\mathrm{n})$. We choose the origin so that $x = 0$ at boundary between the end of p region and the start of the depletion region, i.e. $x_\mathrm{p} = 0$.

Adapted fron Equation 8.4, the electric field in the depletion region's p-type region, using $x_\mathrm{p} = 0$, is

$$E = -\frac{\mathrm{d}\phi}{\mathrm{d}x} = -\frac{e_0 N_\mathrm{a}}{\epsilon\epsilon_0} x.$$

From Equation 9.1, the electrical resitivities in n- and p-type semiconductors are

$$\rho_\mathrm{n} = \frac{1}{e_0 N_\mathrm{d} \mu_\mathrm{e}} \qquad \text{and} \qquad \rho_\mathrm{p} = \frac{1}{e_0 N_\mathrm{a} \mu_\mathrm{h}}.$$

The p-n junction in our problem (with $N_\mathrm{d} \gg N_\mathrm{a}$) consists of two regions: a large p-type region and a small n-type region. We model holes in the p-type region with the time constant

$$\tau_\mathrm{h} \equiv \frac{\epsilon\epsilon_0}{\sigma_\mathrm{E}^{(\mathrm{p})}} = \rho_\mathrm{p}\epsilon\epsilon_0 = \frac{\epsilon\epsilon_0}{e_0 N_\mathrm{a}\mu_\mathrm{h}}.$$

In terms of τ_h, the electric field reads

$$E = -\frac{x}{\mu_\mathrm{h}\tau_\mathrm{h}}.$$

Relating Electrode Voltage and Charge Displacement

We can model the p-n junction as a parallel-plate capacitor, in which the edges of the depletion region on the p- and n-type sides play the role of electrodes.

As derived in Exercise 4.4 (see Equaton 4.6), a charge q moving a distance $\mathrm{d}x$ through a parallel-plate capacitor with capacitance C and electrode separation d causes a voltage drop $\mathrm{d}U$ on the capacitor electrodes given by

$$\mathrm{d}U = \frac{q}{C}\frac{\mathrm{d}x}{d}.$$

We then use $Q = CU \implies \mathrm{d}Q = C\,\mathrm{d}U$ to rewrite the above equation as

$$\mathrm{d}Q = \frac{q\cdot \mathrm{d}x}{d_\mathrm{pn}}.$$

where d_pn is the depletion region width. Interpretation: an electron or hole of charge q moving a distance $\mathrm{d}x$ through depletion region causes a change in voltage $\mathrm{d}Q$ on the detector's electrodes.

Finding Electron Current

The electron velocity through the depletion region is

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = -\mu_\mathrm{e}E = -\mu_\mathrm{e}\cdot\left(-\frac{x}{\mu_\mathrm{h}\tau_\mathrm{h}}\right) = \frac{\mu_\mathrm{e}}{\mu_\mathrm{h}}\frac{x}{\tau_\mathrm{h}}$$

We then separate variables and integrate to find the electrion path $x(t)$ through the depletion region. This reads

$$\int_{x_0}^{x(t)}\frac{\mathrm{d}x'}{x'} = \frac{\mu_\mathrm{e}}{\mu_\mathrm{h}}\int_0^t\frac{\mathrm{d}t'}{\tau_\mathrm{h}} \implies x(t) = x_0 e^{\frac{\mu_\mathrm{e}}{\mu_\mathrm{h}}\frac{t}{\tau_\mathrm{h}}}. \qquad (\text{for } x\in[0, d_\mathrm{pn}])$$

Of course, this expression only applies in the depletion region; an is no longer valid when an electron reaches the n-type electrode at $x = d_\mathrm{pn}$. We then find the time t_e required for electron to reach the n-type electrode by setting $x = d_\mathrm{pn}$. The result is

$$t_\mathrm{e} = \tau_\mathrm{h}\frac{\mu_\mathrm{h}}{\mu_\mathrm{e}}\cdot\ln\frac{d_\mathrm{pn}}{x_0}.$$

With $x(t)$ known we can compute the total charge accumulated on the semiconductor's n-type electrode; this is

$$Q_\mathrm{e}(t) = -\frac{e_0}{d_\mathrm{pn}}\int_0^t\frac{\mathrm{d}x}{\mathrm{d}t'}\,\mathrm{d}t' = +\frac{e_0}{d_\mathrm{pn}}x_0\left(1 - e^{\frac{\mu_\mathrm{e}}{\mu_\mathrm{h}}\frac{t}{\tau_\mathrm{h}}}\right), \qquad (\text{for } t < t_\mathrm{e}). \qquad (9.2)$$

where we have used the minus sign to switch the limits of integration. The corresponding electron current is

$$I_{\mathrm{e}}(t) = \frac{\mathrm{d}Q_{\mathrm{e}}}{\mathrm{d}t} = -\frac{e_0}{d_{\mathrm{pn}}}\frac{x_0}{\tau_{\mathrm{h}}}\frac{\mu_{\mathrm{e}}}{\mu_{\mathrm{h}}} e^{\frac{\mu_{\mathrm{e}}}{\mu_{\mathrm{h}}}\frac{t}{\tau_{\mathrm{h}}}}. \qquad \text{(for } t < t_{\mathrm{e}}\text{)}. \tag{9.3}$$

Finding Hole Current

The hole velocity through the depletion region is

$$v_{\mathrm{h}} = \frac{\mathrm{d}x}{\mathrm{d}t} = \mu_{\mathrm{h}} E = \mu_{\mathrm{h}} \cdot \left(-\frac{x}{\mu_{\mathrm{h}}\tau_{\mathrm{h}}}\right) = -\frac{x}{\tau_{\mathrm{h}}}.$$

As for electrons, we separate variable and solve for the hole path $x(t)$ to get

$$\int_{x_0}^{x(t)} \frac{\mathrm{d}x'}{x'} = -\int_0^t \frac{\mathrm{d}t'}{\tau_{\mathrm{h}}} \implies x(t) = x_0 e^{-\frac{t}{\tau_{\mathrm{h}}}}$$

Note that our simple electric field model predicts holes never actually reach the p region, but only approach asymptotically close (because $E \to 0$ as $x \to 0$?).

Using the hole's time-dependent path $x(t)$, the total charge accumulated on the semiconductor's p-type electrode from holes is then

$$Q_{\mathrm{h}}(t) = +\frac{e_0}{d_{\mathrm{pn}}}\int_0^t \frac{\mathrm{d}x}{\mathrm{d}t'}\,\mathrm{d}t' = -\frac{e_0}{d_{\mathrm{pn}}} x_0 \left(1 - e^{-t/\tau_{\mathrm{h}}}\right).$$

The corresponding hole current is

$$I_{\mathrm{h}}(t) = \frac{\mathrm{d}Q_{\mathrm{h}}}{\mathrm{d}t} = \frac{e_0}{d_{\mathrm{pn}}}\frac{x_0}{\tau_{\mathrm{h}}} e^{-t/\tau_{\mathrm{h}}}. \tag{9.4}$$

9.3.2 Voltage Signal

In practice, we measure the current pulse from the semiconducting detector with a detector with both capacitive and resistive elements; we model this schemitically by considering the semiconductor detector as a current source connected to a parallel capacitor-resistor circuit.

The semiconductor's current is split across the capacitor and resistor according to

$$I = I_{\mathrm{C}} + I_{\mathrm{R}} = C\frac{\mathrm{d}U}{\mathrm{d}t} + \frac{U}{R}, \tag{9.5}$$

where U, which we aim to solve for, denotes the voltage drop across the resistor and capacitor. The total detector current is the sum of the electron and hole contributions, i.e.

$$I = I_{\mathrm{e}} + I_{\mathrm{h}}.$$

Equation 9.5 for U is linear, so its solutions obey the superpostion principle. Out plan is: find the contributions to U from holes and electrons separately, then add the results to get the total measured voltage.

Finding Electron-Induced Voltage

$$I_{\rm e}(t) = -\frac{e_0}{d_{\rm pn}}\frac{x_0}{\tau_{\rm h}}\frac{\mu_{\rm e}}{\mu_{\rm h}}e^{\frac{\mu_{\rm e}}{\mu_{\rm h}}\frac{t}{\tau_{\rm h}}} = C\frac{{\rm d}U}{{\rm d}t} + \frac{U}{R}$$

For shorthand, we define the constants

$$I_0^{\rm e} \equiv -\frac{e_0}{d_{\rm pn}}\frac{x_0}{\tau_{\rm h}}\frac{\mu_{\rm e}}{\mu_{\rm h}} \qquad \text{and} \qquad \tau_{\rm e} \equiv \frac{\mu_{\rm h}}{\mu_{\rm e}}\tau_{\rm h},$$

in terms of which the voltage equation for electrons reads

$$I_{\rm e}(t) = I_0^{\rm e} e^{t/\tau_{\rm e}} = C\frac{{\rm d}U}{{\rm d}t} + \frac{U}{R}. \tag{9.6}$$

This is a nonhomogeneous first-order linear differential equation for U, which will have a general solution of the form $U = U_{\rm h} + U_{\rm p}$, where $U_{\rm h}$ and $U_{\rm p}$ denote the homogeneous and particular solutions, respectively. To save time, we simply quote the solutions:

$$U_{\rm h} = Ae^{-t/RC} \qquad \text{and} \qquad U_{\rm p} = Be^{t/\tau_{\rm e}}.$$

To find the constant B, we substitute $U_{\rm p}$ into Equation 9.6 to get

$$I_0^{\rm e} e^{t/\tau_{\rm e}} = B\left(\frac{C}{\tau_{\rm e}} + \frac{1}{R}\right)e^{t/\tau_{\rm e}} \implies B = I_0^{\rm e}\frac{R}{1 + (RC)/\tau_{\rm e}}. \tag{9.7}$$

The general solution $U(t)$ for electrons then takes the form

$$U_{\rm e}(t) = \begin{cases} Ae^{-\frac{t}{RC}} + Be^{t/\tau_{\rm e}} & t < t_{\rm e} \\ A'e^{-\frac{t}{RC}} & t > t_{\rm e}. \end{cases} \tag{9.8}$$

Note that we consider only one term—the capacitive contribution—for $t > t_{\rm e}$, i.e. when electrons have reached the semiconductor's n-type electrode. In this case the electron current stops, and only stored capacitor energy contributes signal.

We now aim to find the remaining constants in Equation 9.8. First, we apply the condition $U(t = 0) = 0$ (no signal before electron-hole creation) to get $A + B = 0 \implies A = -B$, where B is known from Equation 9.7.

We then find A' from the requirement that $U(t)$ is continuous at $t = t_{\rm e}$; this leads to

$$B\left(-e^{-\frac{t_{\rm e}}{RC}} + e^{t_{\rm e}/\tau_{\rm e}}\right) = A'e^{-\frac{t_{\rm e}}{RC}} \implies A' = Be^{\frac{t_{\rm e}}{RC}}\left(e^{t_{\rm e}/\tau_{\rm e}} - e^{-\frac{t_{\rm e}}{RC}}\right)$$

We then substitute in constants to get the final result (for electrons):

$$U_{\rm e}(t) = \frac{I_0^{\rm e}R}{1 + (RC)/\tau_{\rm e}}\begin{cases} e^{t/\tau_{\rm e}} - e^{-\frac{t}{RC}} & t < t_{\rm e} \\ \left(e^{t_{\rm e}/\tau_{\rm e}} - e^{-\frac{t_{\rm e}}{RC}}\right)e^{-\frac{(t-t_{\rm e})}{RC}} & t > t_{\rm e}. \end{cases} \tag{9.9}$$

Finding Hole-Induced Voltage

For holes, using Equation 9.4 to express $I_{\rm h}$, Equation 9.5 for U reads

$$I_{\rm h}(t) = \frac{e_0}{d_{\rm pn}}\frac{x_0}{\tau_{\rm h}}e^{-t/\tau_{\rm h}} = C\frac{{\rm d}U}{{\rm d}t} + \frac{U}{R}.$$

For shorthand, we define the constant

$$I_0^{\mathrm{h}} \equiv \frac{e_0}{d_{\mathrm{pn}}} \frac{x_0}{\tau_{\mathrm{h}}},$$

in terms of which the voltage equation for holes reads

$$I_{\mathrm{h}}(t) = I_0^{\mathrm{h}} e^{-t/\tau_{\mathrm{h}}} = C\frac{\mathrm{d}U}{\mathrm{d}t} + \frac{U}{R}. \tag{9.10}$$

We again have a nonhomogeneous first-order linear differential equation for U, which will have a general solution of the form $U = U_{\mathrm{h}} + U_{\mathrm{p}}$. To save time, we simply quote the solutions:

$$U_{\mathrm{h}} = Ae^{-t/RC} \qquad \text{and} \qquad U_{\mathrm{p}} = Be^{-t/\tau_{\mathrm{h}}},$$

the main difference from the electron case being the minus in the particular solution's exponent. To find the constant B, we substitute U_{p} into Equation 9.10 to get

$$I_0^{\mathrm{h}} e^{-t/\tau_{\mathrm{h}}} = B\left(-\frac{C}{\tau_{\mathrm{h}}} + \frac{1}{R}\right) e^{-t/\tau_{\mathrm{h}}} \implies B = I_0^{\mathrm{h}} \frac{R}{1-(RC)/\tau_{\mathrm{h}}}. \tag{9.11}$$

The general solution $U(t)$ for electrons then takes the form

$$U_{\mathrm{h}}(t) = Ae^{-\frac{t}{RC}} + Be^{-t/\tau_{\mathrm{h}}}. \tag{9.12}$$

Unlike for the electron signal U_{e} in Equation 9.7 we don't need a piecewise solution, since the hole current (Equation 9.4) is defined for all times. We find the constant A from condition $U(t=0) = 0$ (no signal before electron-hole creation) to get $A + B = 0 \implies A = -B$, where B is known from Equation 9.11. The complete hole contribution U_{h} to the detector signal is then

$$U_{\mathrm{h}}(t) = Ae^{-\frac{t}{RC}} + Be^{-t/\tau_{\mathrm{h}}} = \frac{I_0^{\mathrm{h}} R}{1-(RC)/\tau_{\mathrm{h}}} \left(e^{-t/\tau_{\mathrm{h}}} - e^{-\frac{t}{RC}}\right). \tag{9.13}$$

The complete detector signal is then the sum

$$U(t) = U_{\mathrm{e}}(t) + U_{\mathrm{h}}(t),$$

where U_{e} and U_{h} are given by Equations 9.9 and 9.13, respectively, by

$$U_{\mathrm{e}}(t) = \frac{I_0^{\mathrm{e}} R}{1+(RC)/\tau_{\mathrm{e}}} \begin{cases} e^{t/\tau_{\mathrm{e}}} - e^{-\frac{t}{RC}} & t < t_{\mathrm{e}} \\ \left(e^{t_{\mathrm{e}}/\tau_{\mathrm{e}}} - e^{-\frac{t_{\mathrm{e}}}{RC}}\right) e^{-\frac{(t-t_{\mathrm{e}})}{RC}} & t > t_{\mathrm{e}}. \end{cases}$$

$$U_{\mathrm{h}}(t) = \frac{I_0^{\mathrm{h}} R}{1-(RC)/\tau_{\mathrm{h}}} \left(e^{-t/\tau_{\mathrm{h}}} - e^{-\frac{t}{RC}}\right),$$

where the constants I_0^{h}, I_0^{e} and τ_{e} are given in terms of the detector's parameters by

$$I_0^{\mathrm{h}} = \frac{e_0}{d_{\mathrm{pn}}} \frac{x_0}{\tau_{\mathrm{h}}}, \qquad I_0^{\mathrm{e}} = -\frac{e_0}{d_{\mathrm{pn}}} \frac{x_0}{\tau_{\mathrm{h}}} \frac{\mu_{\mathrm{e}}}{\mu_{\mathrm{h}}}, \qquad \tau_{\mathrm{e}} = \frac{\mu_{\mathrm{h}}}{\mu_{\mathrm{e}}} \tau_{\mathrm{h}}.$$

9.3.3 Limit Cases of Electron Signal

For review, the general electron-induced signal reads

$$U_{\mathrm{e}}(t) = \frac{I_0^{\mathrm{e}} R}{1 + (RC)/\tau_{\mathrm{e}}} \begin{cases} e^{t/\tau_{\mathrm{e}}} - e^{-\frac{t}{RC}} & t < t_{\mathrm{e}} \\ \left(e^{t_{\mathrm{e}}/\tau_{\mathrm{e}}} - e^{-\frac{t_{\mathrm{e}}}{RC}}\right) e^{-\frac{(t - t_{\mathrm{e}})}{RC}} & t > t_{\mathrm{e}}. \end{cases}$$

In the limit case $RC \ll \tau_{\mathrm{e}}$, the electron signal simplifies to

$$U_{\mathrm{e}}(t) = I_0^{\mathrm{e}} R \begin{cases} e^{t/\tau_{\mathrm{e}}} - e^{-\frac{t}{RC}} & t < t_{\mathrm{e}} \\ \left(e^{t_{\mathrm{e}}/\tau_{\mathrm{e}}} - e^{-\frac{t_{\mathrm{e}}}{RC}}\right) e^{-\frac{(t - t_{\mathrm{e}})}{RC}} & t > t_{\mathrm{e}}. \end{cases} \qquad (\text{if } RC \ll \tau_{\mathrm{e}})$$

In this case the signal U_{e} measured at the detector follows the raw detector signal. A detector working in this regime is called a *voltage-sensitive* detector.

In the limit case $RC \gg \tau_{\mathrm{e}}$, we first make the auxiliary steps

$$\frac{I_0^{\mathrm{e}} R}{1 + (RC)/\tau_{\mathrm{e}}} = \frac{(I_0^{\mathrm{e}} \tau_{\mathrm{e}})/C}{1 + \tau_{\mathrm{e}}/(RC)} \approx \frac{I_0^{\mathrm{e}} \tau_{\mathrm{e}}}{C} \qquad \text{and} \qquad e^{-\frac{t_{\mathrm{e}}}{RC}} \approx 1,$$

in terms of which the electron signal simplifies to

$$U_{\mathrm{e}}(t) = \frac{I_0^{\mathrm{e}} \tau_{\mathrm{e}}}{C} \begin{cases} e^{t/\tau_{\mathrm{e}}} - e^{-\frac{t}{RC}} & t < t_{\mathrm{e}} \\ \left(e^{t_{\mathrm{e}}/\tau_{\mathrm{e}}} - 1\right) e^{-\frac{(t - t_{\mathrm{e}})}{RC}} & t > t_{\mathrm{e}} \end{cases} \qquad (\text{for } RC \gg \tau_{\mathrm{e}})$$

We will be interested in the regime $t > t_{\mathrm{e}}$.

To proceed, we recall Equation 9.2 for electron charge, which for review reads

$$Q_{\mathrm{e}}(t) = \frac{e_0}{d_{\mathrm{pn}}} x_0 \left(1 - e^{\frac{\mu_{\mathrm{e}}}{\mu_{\mathrm{h}}} \frac{t}{\tau_{\mathrm{h}}}}\right) = I_0^{\mathrm{e}} \tau_{\mathrm{e}} (e^{t/\tau_{\mathrm{e}}} - 1), \qquad (\text{for } t < t_{\mathrm{e}}).$$

The total charge reaching the detector electrodes, i.e. $Q_{\mathrm{e}}(t = t_{\mathrm{e}})$, is

$$Q_{\mathrm{e}}(t_{\mathrm{e}}) = I_0^{\mathrm{e}} \tau_{\mathrm{e}} (e^{t_{\mathrm{e}}/\tau_{\mathrm{e}}} - 1),$$

in terms of which $U_{\mathrm{e}}(t)$, for $t > t_{\mathrm{e}}$, reads

$$U_{\mathrm{e}}(t) = \frac{I_0^{\mathrm{e}} \tau_{\mathrm{e}}}{C} \left(e^{t_{\mathrm{e}}/\tau_{\mathrm{e}}} - 1\right) e^{-\frac{(t - t_{\mathrm{e}})}{RC}} = \frac{Q_{\mathrm{e}}(t_{\mathrm{e}})}{C} e^{-\frac{(t - t_{\mathrm{e}})}{RC}} \qquad (\text{for } RC \gg \tau_{\mathrm{e}})$$

A detector working in this regime is called a charge-sensitive detector.

10 Tenth Exercise Set

10.1 Theory: Position-Sensitive Semiconducting Detectors

We begin with a simple example: a silicon strip of length L made by joining p- and n-type doped semiconductors. As discussed in the past few exercises, a particle incident in the strip's depletion region frees ion-hole pairs which are accelerated to the semiconductor faces by the depletion region's electric field.

The charge accumulated at the top and bottom electrode encodes the energy deposited by an incident particle.

Assume the electrodes are wired to measured a particle's deposited energy such that

$$U_{\text{top}} \propto E_{\text{dep}} \frac{x}{L} \qquad \text{and} \qquad U_{\text{bottom}} \propto E_{\text{dep}},$$

where L is the length of the strip and x is the position along the strip. We then divide the top and bottom electrode signals to get

$$\frac{U_{\text{top}}}{U_{\text{bottom}}} = \frac{x}{L}.$$

The resulting signal is proportional to position x of an incident particle along length of strip. Resolutions for a simple strip configuration are of the order $\sigma_x \sim 300\,\mu\text{m}$.

Micro-Strip Detector

A micro-strip detector uses grid of semiconducting strips separated by the *pitch distance* p, which is typically of the order $p \sim 20$ to $100\,\mu\text{m}$; we use x_i to denote the position of the i-th strip.

We then detect the position of an ionizing particle in the strip grid based on which microstrip returns the strongest signal (or, more accurately, with a weighted average of strips in the neighborhood of the strongest signal).

Quoting from the results derived in lecture, the resolution σ_x when measuring particle position with a single strip is

$$\sigma_x = \frac{p}{\sqrt{12}}.$$

where p is the strip pitch (i.e. the spacing between neighboring strips).

For better resolution, we measure position using multiple strips. We first define the charge-weighted average position

$$\overline{x} = \frac{\sum_i Q_i x_i}{\sum_i Q_i},$$

where Q_i is the measured charge on the i-th strip and x_i is the position of the i-th strip. In this case, without derivation, the resolution of $\overline{x}$ obeys

$$\sigma_{\overline{x}}^2 \propto p^2 \frac{\sum_j \sigma_{Q_j}^2}{\left(\sum_i Q_i\right)^2} = p^2 \frac{(\text{noise})^2}{(\text{signal})^2} = \frac{p^2}{\text{SNR}^2},$$

where $\sigma_{Q_j}^2$ is the resolution of the charge measurement from the j-th strip. As might be expected, resolution improves with decreasing strip separation p and increaseing signal to noise ratio SNR.

10.2 Theory: Statistics

10.2.1 Deriving the Poisson Distribution

Show that the number of independent random events in a given time interval T, for which the probability per unit time is λ, is Poisson-distributed.

We first separate the time interval T into N small segments of length $\tau = T/N$, where we assume N is large enough that $\gamma \ll 1/\lambda$. For large N we then have $\lambda\tau \ll 1$ (the probability of a random event occuring in the interval τ is very small) and can neglect higher-order terms, i.e. $(\lambda\tau)^m \to 0$ for $m \geq 2$ (the probability of 2 or more event occuring in the interval τ is negligible).

We will derive the Poisson distribution from the binomial distribution, which states that the probability of n events occurring over the course of N trials, where the probability of an event occurring in a single trial is p, is given by

$$P(n|N,p) = \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}.$$

We then apply the binomial distribution to the above-described random events, interpreting a "trial" as a single interval of time $\tau = T/N$, while the probability of an event occuring during a single trial is $\lambda\tau = (\lambda T)/N$. With $p \to (\lambda T)/N$ the binomial distribution becomes

$$P(n|N,\lambda,T) = \frac{N!}{n!(N-n)!} \left(\frac{\lambda T}{N}\right)^n \left(1 - \frac{\lambda T}{N}\right)^{N-n} \tag{10.1}$$

We will then apply the limit $N \to \infty$, but first make the auxiliary calculations

$$\begin{aligned}
\lim_{N\to\infty} \frac{N!}{(N-n)! \cdot N^n} &= \lim_{N\to\infty} \frac{N \cdot (N-1) \cdots (N-n+1)}{N^n} \\
&= \lim_{N\to\infty} \frac{N}{N} \cdot \left(\frac{N-1}{N}\right) \cdots \left(\frac{N-n+1}{N}\right) \\
&= 1^n = 1,
\end{aligned}$$

and

$$\lim_{N\to\infty} \left(1 - \frac{\lambda T}{N}\right)^N = e^{-\lambda T}$$

Using these identities, in the limit $N \to \infty$, Equation 10.1 becomes

$$P(n|\lambda,T) = \frac{(\lambda T)^n}{n!} e^{-\lambda T}.$$

Finally, we define the probability $\mu \equiv \lambda T$ and get the desired Poisson distribution

$$P(n|\mu) = \frac{\mu^n e^{-\mu}}{n!}.$$

10.2.2 Statistical Significance

An experiment measures signal events with frequency s and background events with frequency b. Determine the necessary data acquisition time t such that the experiment measures signal with a statistical significance of n standard deviations.

We define the statistical significance of signal/background classification problems as

$$n = \frac{N_{\mathrm{s}}}{\sigma_{\mathrm{b}}}$$

where N_{s} is the number of measured signal events and σ_{b} is the fluctuation of the background events about their mean value. In this sense, statistical significance is the number of measured signal events per unit background fluctuation.

When the number of background events is large, i.e. $N_{\mathrm{b}} \gg 1$, we assume the background events obey Poisson statistics, in which case the background fluctuation for an experiment measuring N_{b} background events is

$$\sigma_{\mathrm{b}} = \sqrt{N_{\mathrm{b}}}.$$

In the given problem the signal and background rates s and b are known, so the number of signal and background events measured in an experiment of duration t is

$$N_{\mathrm{s}} = st \qquad \text{and} \qquad N_{\mathrm{b}} = bt.$$

The experiment's statistical significance for signal detection is then

$$n = \frac{N_{\mathrm{s}}}{\sigma_{\mathrm{b}}} = \frac{N_{\mathrm{s}}}{\sqrt{N_{\mathrm{b}}}} = \frac{st}{\sqrt{bt}},$$

and so the required experiment time t to reach a significance of n background standard deviation is

$$t = n^2 \frac{b}{s^2}.$$

11 Eleventh Exercise Set

11.1 Statistical Significance in a Radioactive Decay Experiment

In a hypothetical experiment designed as an exercise in statistical significance, a closed box potentially *contains a radioactive source, but the person carrying out the experiment does not know ahead of time if the source is in the box or not. The experiment's background activity is known to be* b *(i.e.* b *background events per unit time) and the source activity, if the source where in the box, is known to be* s*.*

By counting radiactive decay events over a time period t *with a Geiger counter placed immediately outside the box, determine the statistical significance with which can you claim the presence of the source in the box.*

Let N denote the total number of events—signal or background—measured over the course of the experiment, so that

$$N = N_{\text{sig}} + N_{\text{bg}}$$

From the known background activity, we expect

$$\overline{N}_{\text{bg}} = bt.$$

Background activity is well approximated as a random Poisson process, so the experiment's background fluctuations will be

$$\sigma_{\text{bg}} = \sqrt{N_{\text{bg}}}.$$

If the box did contain the radioactive source, the expected number of signal events over a time period t would be

$$st = \overline{N}_{\text{sig}}$$

The signal's statistical significance is then

$$n = \frac{\overline{N}_{\text{sig}}}{\sigma_{\text{b}}} = \frac{\overline{N}_{\text{sig}}}{\sqrt{\overline{N}_{\text{bg}}}} = \frac{s \cdot t}{\sqrt{b \cdot t}} = \frac{s}{\sqrt{b}}\sqrt{t}.$$

Under the hypothesis "only background" or "no source in box" we expect to measure N *or more* events (over a large number of experiments, each of time t) with the probability

$$P = 1 - \Phi(n\sigma_{\text{bg}}),$$

where Φ is the standard normal distribution's cumulative distribution function.

Interpretation: the larger the value of n, the smaller the value of P, and the less likely it is to measure N or more events under the hypotheses "background only" (because of the likely contribution of additional signal events, which increase the total event count).

11.2 Lower Bound on Decay Time at a 90% Confidence Level

In a search for hypothesized neutrino-less double beta decay, an experiment observes a 50 g sample of selenium-82 over a period of 100 days and does not measure any signal events. The experiment's detection efficiency for the theorized double-beta signal process is 20%. At a 90% confidence level, place a lower bound on selenium-82's decay time for the double-beta decay.

We first write decay time τ in terms of decay constant λ as $\tau = 1/\lambda$. Beta decay is a random process, and the probability for n decay events occuring over a time interval T is distributed according to the Poisson distribution

$$P(n|\lambda, T) = \frac{(\lambda T)^n e^{-\lambda T}}{n!}.$$

The probability of measuring $n = 0$ zero events is then

$$P(0|\lambda, T) = e^{-\lambda T}.$$

Alternatively, we can view this result for $P(0|\lambda, T)$ as a probability distribution for the decay constant λ (in the case for which the experiment observes $n = 0$ signal events). We then assume the decay constant λ is distributed according to

$$\frac{\mathrm{d}P}{\mathrm{d}\lambda} = Te^{-\lambda T},$$

where we have included the constant T for normalization.

The problem asks for a lower bound τ_0 on selenium-82's decay time for double beta decay. From the inverse relationship $\lambda = 1/\tau$, a lower bound on τ is equivalent to an upper bound on λ. Based on the distribution $\frac{\mathrm{d}P}{\mathrm{d}\lambda} = Te^{-\lambda T}$, we are thus interested in the probability that $\lambda < \lambda_0$, which is

$$P(\lambda < \lambda_0) = \int_0^{\lambda_0} Te^{-\lambda T} \, \mathrm{d}\lambda = 1 - e^{-\lambda_0 T}.$$

At a 90 percent confidence level for λ_0 as a lower bound, we aim to solve

$$0.9 = P(\lambda < \lambda_0) = 1 - e^{-\lambda_0 T} \implies \lambda_0 = -\frac{1}{T} \ln(1 - 0.9).$$

With some well-chosen logarithm properties, λ_0 simplifies to

$$\lambda_0 = -\frac{1}{T} \ln(1 - 0.9) = -\frac{1}{T} \ln 0.1 = \frac{1}{T} \ln(0.1)^{-1} = \frac{\ln 10}{T}.$$

Substituting in numerical values and accounting for imperfect decection efficiency $\eta = 0.2$, we find

$$\lambda_0 = \frac{\ln 10}{\eta \cdot T} = \frac{\ln 10}{0.2 \cdot 100 \, \mathrm{day}} = 0.115 \, \mathrm{day}^{-1}.$$

Interpretation: if the experiment does not measure double beta decay event in the time T, then we can, with confidence level $P(\lambda < \lambda_0) = 0.9$, claim that the true decay constant λ is less than λ_0.

Technicality: the just-derived λ_0 is the hypothetical decay constant for the experiment's entire 50 g sample of selenium-82. A more physically relevant quantity is the decay constant for a single nucleus, e.g. $\lambda_0^{(\mathrm{nuc})}$.

The total number of nuclei in the experiment's $m = 50\,\mathrm{g}$ sample is

$$N_{\mathrm{nuclei}} = \frac{N_A}{M} \cdot m = \frac{N_A}{82\,\mathrm{g\,mol^{-1}}} \cdot 50\,\mathrm{g} = 3.67 \cdot 10^{23},$$

and so the decay constant for a single Se-82 nucleus is

$$\lambda_0^{(\mathrm{nuc})} = \frac{\lambda_0}{N_{\mathrm{nuclei}}} = \frac{115\,\mathrm{day}^{-1}}{3.67 \cdot 10^{23}} = 3.13 \cdot 10^{-25}\,\mathrm{day}^{-1},$$

and the corresponding decay time is the reciprocal value

$$\tau_0 = \frac{1}{\lambda_0^{(\mathrm{nuc})}} = 8.75 \cdot 10^{21}\,\mathrm{year}.$$

For the lower bound τ_0 to be well-defined, we must also quote the confidence level, so the experiment's result for the lower bound on the double beta decay constant τ in Se-82 should read

$$\tau > \tau_0 = 8.75 \cdot 10^{21}\,\mathrm{year}, \text{ at the confidence level CL} = 0.9.$$

Note that this lower bound on decay time is enormous—as this result suggests, neutrino-less double beta decay has never been observed.

11.3 Signal Dynamics in a Scintillating Detector

Consider a scintillating detector consisting of a scintillator and photomultiplier tube. Derive the time-dependent current and voltage signals $I(t)$ and $U(t)$ registed by detector after an incident particle causes a shower of scintillation photons.

For orientation: a scintillating detector consists of a scintillator and a photodetector—we will consider a photomultiplier tube (PMT). An incident particle (which we wish to detect) hits the scintillator, depositing energy and producing scintillation photons. Scintillation photons reach photomultiplier tube's cathode and produce photoelectrons. Photoelectrons are accelerated and multiplied (by a factor of the order 10^6) through a dynode chain and incident on a collecting anode.

Just like in Exercise 9.3, in which we analyzed as semiconductor's signal dynamics, we measure the current with a detector containing both capacitive and resistive elements. We model this schematically by considering the PMT as a current source connected to a parallel capacitor-resistor circuit.

For review from Exercise 9.3, the PMT current I is split across the capacitor and resistor according to

$$I = I_{\mathrm{C}} + I_{\mathrm{R}} = C\frac{\mathrm{d}U}{\mathrm{d}t} + \frac{U}{R},$$

where U, which we aim to solve for, is the voltage across the resistor and capacitor.

The number of scintillation photons produced in the scintillator in response to an incident particle passing through at $t = 0$ takes the form

$$N(t) = N_0 e^{-t/\tau},$$

where the response time τ is a property of the scintillator. For orientation, $\tau \approx 230\,\text{ns}$ for NaI and $\tau \approx 1\,\text{ns}$ for BaF.

For simplicity, we will assume the PMT response time is much faster than the scintillator response time τ, meaning the PMT current follows the flux of scintillation photons incident on the PMT cathode. In this case the PMT current I follows the same functional form as N, i.e.

$$I(t) = I_0 e^{-t/\tau}.$$

Quoted: This approximation is often acceptable in practice, especially for scintillators with large τ (e.g $\tau \sim 100\,\text{ns}$), since typical photodetectors are very fast.

Our first step is to estimate the PMT current amplitude I_0. First, the total charge Q reaching the PMT anode for a scintillator emitting $\frac{\mathrm{d}N}{\mathrm{d}E}$ photons per unit energy deposited in the scintillator is

$$Q = \frac{\mathrm{d}N}{\mathrm{d}E} \cdot E_{\text{dep}} \cdot \eta \cdot M \cdot e_0,$$

where E_{dep} is the energy deposited in the scintillator by an incident particle, η is the photomultiplier's efficiency (of the order 10% to 40%) and M is the photomultiplier's multiplication factor. Typical values of $\frac{\mathrm{d}N}{\mathrm{d}E}$ are of the order 10 to $30\,\text{keV}^{-1}$.

Writing $Y \equiv \frac{\mathrm{d}N}{\mathrm{d}E}$ for shorthand, the charge registered at the PMT anode is

$$Q = Y \cdot E_{\text{dep}} \cdot \eta \cdot M \cdot e_0,$$

We then find PMT anode current amplitude I_0 from the integral

$$Q = \int_0^\infty I(t)\,\mathrm{d}t = \int_0^\infty I_0 e^{-t/\tau}\,\mathrm{d}t = -\left[\frac{I_0}{\tau} e^{-t/\tau}\right]_0^\infty = I_0 \tau.$$

We then solve for I_0 to get

$$I_0 = \frac{Q}{\tau} = \frac{e_0 M \eta E_{\text{dep}} Y}{\tau}$$

Assuming efficiency $\eta = 0.1$, deposited energy $E_{\text{dep}} = 1\,\text{MeV}$, scintillation photons freed per unit deposited energy $Y = 10$ to $30\,\text{keV}^{-1}$, and PMT multiplication factor and response time $M = 10^6$ and $\tau = 0.5\,\mu\text{s}$, the PMT current amplitude is

$$Q \sim (1.5 - 5.0) \cdot 10^{-10}\,\text{C} \implies I_0 = \frac{Q}{\tau} = (3 - 10) \cdot 10^{-4}\,\text{A}.$$

We will assume we register the PMT signal using a simple parallel capacitor-resistor wiring, where the PMT current I is split across the capacitor and resistor to give

$$I = I_{\text{C}} + I_{\text{R}} \tag{11.1}$$

Letting U denote the voltage across the resistor (essentially voltage on oscilloscope, if we model oscilloscope input impedance with a resistor), the currents through the resistor and capacitor are

$$I_{\mathrm{C}}(t) = C\frac{\mathrm{d}U}{\mathrm{d}t} \qquad \text{and} \qquad I_{\mathrm{R}}(t) = \frac{U(t)}{R}.$$

We then substitute I_{C} and I_{R} into Equation 11.1 to get the following differential equation for $U(t)$:

$$I_0 e^{-t/\tau} = C\frac{\mathrm{d}U}{\mathrm{d}t} + \frac{U(t)}{R}.$$

To save time, we reuse the earlier semiconductor signal results from Section 9.3.2, in which we would again find the equation's particular and homogeneous solutions. Without derivation, the result is

$$U(t) = \frac{I_0 R}{1 - (RC)/\tau}\left(e^{-t/\tau} - e^{-\frac{t}{RC}}\right). \tag{11.2}$$

We will consider Equation 11.2 in two limit cases:

1. For $R = 1\,\mathrm{M}\Omega$ we assume $RC \gg \tau$, and for $t \gg \tau$ the oscilloscope voltage reads

$$U(t) \approx \frac{I_0\tau}{C}e^{-t/RC} = \frac{Q}{C}e^{-t/(RC)} \qquad \text{(if } RC \gg \tau\text{)}.$$

   Lesson: the oscilloscope voltage is proportional to the charge accumulated on the PMT anode. This large RC regime is called charge-sensitive mode.

2. For $R = 50\,\Omega$ we assume $RC \ll \tau$, and for $t \gg RC$ the oscilloscope signal reads

$$U(t) \approx RI_0 e^{-t/\tau} = RI(t) \qquad \text{(if } RC \ll \tau\text{)}.$$

   Lesson: the oscilloscope voltage is proportional to the current from the PMT anode. This small RC regime is called current-sensitive mode.

12 Twelfth Exercise Set

12.1 Efficiency of a CsI(Tl) Scintillator

A CsI(Tl) *emits* $\frac{\mathrm{d}N}{\mathrm{d}E} = 6.5 \cdot 10^4\,\mathrm{MeV}^{-1}$ *scintillation photons per unit absorbed energy, with a photon wavelength* $\lambda = 565\,\mathrm{nm}$. *Estimate the scintillator's efficiency for an incident particle depositing* $E_{\mathrm{dep}} = 1\,\mathrm{MeV}$ *of energy in the scintillator.*

The scintillator's efficiency is given by

$$\eta = \frac{E_{\mathrm{scint}}}{E_{\mathrm{dep}}},$$

where E_{dep} is the energy deposited by an incident particle and E_{scint} is the energy of all emitted scintillation photons. We find the total scintillation photon energy from

$$E_{\mathrm{scint}} = N_{\mathrm{scint}} E_{\gamma},$$

where E_{γ} is the energy of a single photon and N_{scint} is the total number of emitted photons. Using the general quantum relationship $E_{\gamma} = h\nu$ for photon energy, the single-photon energy E_{γ} is

$$E_{\gamma} = h\nu = \frac{hc}{\lambda} = \frac{1240\,\mathrm{eV\,nm}}{565\,\mathrm{nm}} \approx 2.2\,\mathrm{eV}.$$

The energy of all N_{scint} scintillation photons is then simply

$$E_{\mathrm{scint}} = N_{\mathrm{scint}} \cdot E_{\gamma} = \left(\frac{\mathrm{d}N}{\mathrm{d}E} \cdot E_{\mathrm{dep}}\right) \cdot E_{\gamma}.$$

For an incident particle depositing energy $E_{\mathrm{dep}} = 1\,\mathrm{MeV}$, the total scintillation photon energy is then

$$E_{\mathrm{scint}} = \left(\frac{\mathrm{d}N}{\mathrm{d}E} \cdot E_{\mathrm{dep}}\right) \cdot E_{\gamma} = (6.5 \cdot 10^4\,\mathrm{MeV}^{-1}) \cdot (1\,\mathrm{MeV}) \cdot (2.2\,\mathrm{eV}) \approx 143\,\mathrm{keV},$$

and so the scintillator's efficiency is

$$\eta = \frac{E_{\mathrm{scint}}}{E_{\mathrm{dep}}} = \frac{0.143\,\mathrm{MeV}}{1\,\mathrm{MeV}} \approx 0.14,$$

which is fairly typical for a moden scintillator.

12.2 Theory: Effect of Fluctuations on Photomultiplier Amplification

Situation: incident particle emits scintillation photons, which produce n photoelectrons in PMT cathode. Each of initial photoelectrons is accelerated through the PMT dynode chain, producing a large number of secondary electrons at the PMT anode. Let X_i denote the (large) number of secondary electrons reaching the PMT anode as a result of the i-th initial cathode photoelectron.

Let S_n be the sum of all secondary electrons at the PMT anode when an incident particle frees n cathode photoelectrons; this is

$$S_n = \sum_{i=1}^{n} X_i. \tag{12.1}$$

The expected value $\langle X \rangle$ of the single-electron multiplication factors is the photomultiplier's multiplication factor.

Both n and the X are random variables. We assume:

- the number n of initial cathode photoelectrons is distributed according to a Poisson distribution with mean λ, i.e. $\langle n \rangle = \lambda$, and
- the single-photon multiplication factors X_i are distributed according to an arbitrary (not specified) probability distribution.

Our goals are to:

1. show that $\langle S \rangle = \lambda \cdot \langle X \rangle$, and
2. find σ_S^2—the fluctuations in total number multiplied electrons at the PMT anode.

Finding the Expected Value $\langle S \rangle$

We first aim to show that $\langle S \rangle = \lambda \langle X \rangle$. From first principles, $\langle S \rangle$ is

$$\langle S \rangle = \sum_{n=0}^{\infty} P_n \cdot \langle S_n \rangle$$

where P_n is the probability that an incident particle frees n initial (pre-multiplication cathode) photoelectrons, while S_n is the expected number of post-multiplication anode photoelectrons resulting from the n cathode photoelectrons.

In other words, $\langle S \rangle$ is just a probability weighted average of the $\langle S_n \rangle$ of all possible n, (which we know are distributed according to a Poisson distribution with mean λ).

From Equation 12.1, the expectation $\langle S_n \rangle$ is

$$\langle S_n \rangle = \left\langle \sum_{i=0}^{n} X_i \right\rangle = \sum_{i=0}^{n} \langle X \rangle = n \langle X \rangle .$$

We then substitute $\langle S_n \rangle$ into $\langle S \rangle$ and get

$$\langle S \rangle = \sum_{n=0}^{\infty} P_n \langle S_n \rangle = \langle X \rangle \sum_{n=0}^{\infty} n P_n = \langle X \rangle \langle n \rangle = \lambda \langle X \rangle . \tag{12.2}$$

where we have used the definition $\sum_{n=0}^{\infty} n P_n \equiv \langle n \rangle$ and the given mean value $\langle n \rangle = \lambda$.

Finding Fluctuations σ_S^2

We will find σ_S^2 from the general identity

$$\sigma_S^2 = \langle S^2 \rangle - \langle S \rangle^2 \tag{12.3}$$

The expectation $\langle S \rangle$ is known from Equation 12.2 above. Meanwhile, we find $\langle S^2 \rangle$ from the definition

$$\langle S^2 \rangle = \sum_{n=0}^{\infty} P_n \langle S_n^2 \rangle . \tag{12.4}$$

For review from Equation 12.1, S_n is given by

$$S_n = \sum_{i=1}^{n} X_i \implies S_n^2 = \left(\sum_{i=1}^{n} X_i \right) \cdot \left(\sum_{j=1}^{n} X_j \right) .$$

From here we find $\langle S_n \rangle^2$, by definition, according to

$$\begin{aligned} \langle S_n^2 \rangle &= \left\langle \left(\sum_{i}^{n} X_i \right) \left(\sum_{j}^{n} X_j \right) \right\rangle = \left\langle \sum_{i} X_i^2 + \sum_{i \neq j} X_i X_j \right\rangle \\ &= \sum_{i} \langle X_i^2 \rangle + \sum_{i \neq j} \langle X_i X_j \rangle \\ &= n \langle X^2 \rangle + n(n-1) \langle X \rangle^2 , \end{aligned}$$

where we have split the initial product of sums into n diagonal and $n(n-1)$ off-diagonal elements (and assumed the X_i are independent to get $\langle X_i X_j \rangle = \langle X^2 \rangle$?).

We then substitute the result for $\langle S_n \rangle^2$ into Equation 12.4 to get

$$\begin{aligned} \langle S^2 \rangle &= \sum_{n=0}^{\infty} P_n \langle S_n^2 \rangle = \langle X^2 \rangle \sum_{n=0}^{\infty} n P_n + \langle X \rangle^2 \sum_{n=0}^{\infty} n(n-1) P_n \\ &= \langle X^2 \rangle \langle n \rangle + \langle X \rangle^2 \Big(\langle n^2 \rangle - \langle n \rangle \Big). \end{aligned}$$

Finally, we substite $\langle S^2 \rangle$ into Equation 12.3 to get

$$\begin{aligned} \sigma_S^2 = \langle S^2 \rangle - \langle S \rangle^2 &= \Big[\langle X^2 \rangle \langle n \rangle + \langle X \rangle^2 \Big(\langle n^2 \rangle - \langle n \rangle \Big) \Big] - \Big(\langle n \rangle \langle X \rangle \Big)^2 \\ &= \langle X^2 \rangle \langle n \rangle + \langle X \rangle^2 \Big[\langle n^2 \rangle - \langle n \rangle - \langle n \rangle^2 \Big]. \end{aligned} \tag{12.5}$$

Recall n is Poisson-distributed, and thus obeys

$$\langle n \rangle = \lambda, \quad \text{and} \quad \sigma_n^2 = \lambda \implies \sigma_n^2 = \langle n^2 \rangle - \langle n \rangle^2 = \lambda.$$

We then substitute $\langle n^2 \rangle - \langle n \rangle^2 = \lambda$ and $\langle n \rangle = \lambda$ into Equation 12.5, which simplifies considerably to

$$\sigma_S^2 = \langle X^2 \rangle \lambda + \langle X \rangle^2 \cdot (\lambda - \lambda) = \lambda \langle X^2 \rangle .$$

Finally, we write σ_S^2 in the equivalent form

$$\sigma_S^2 = \lambda \langle X^2 \rangle = \lambda \left(\sigma_X^2 - \langle X \rangle^2 \right) = \lambda \langle X^2 \rangle \left(1 + \frac{\sigma_X^2}{\langle X \rangle^2} \right)$$

Interpretation: the PMT amplification factor $\langle X^2 \rangle$ were a well-defined, non-fluctuating variable, i.e. if $\sigma_X^2 = 0$, then S would fluctuate only as $\sigma_S^2 = \lambda \langle X^2 \rangle$. But because the single-electron multiplication factors X_i fluctuate, σ_S^2 is larger by a factor

$$F \equiv 1 + \frac{\sigma_X^2}{\langle X \rangle^2},$$

which is called the photomultiplier's excess noise factor.

Example: Exponentially-Distributed Multiplication Factor

So far we did not specify a distribution for the multiplication factor X. However, in many photomultipliers, X is exponentially distributed as

$$f(x) = ae^{-ax},$$

from which we can compute the expectation values

$$\langle X \rangle = \int_0^\infty x f(x)\,\mathrm{d}x = \frac{1}{a} \qquad \text{and} \qquad \langle X^2 \rangle = \int_0^\infty x^2 f(x)\,\mathrm{d}x = \frac{2}{a^2}.$$

Using $\langle X \rangle$ and $\langle X^2 \rangle$ we have $\sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2 = \frac{1}{a^2}$ and thus

$$F = 1 + \frac{\sigma_X^2}{\langle X \rangle^2} = 1 + \frac{a^2}{a^2} = 2 \implies \sigma_S^2 = 2\lambda \langle X^2 \rangle.$$

Lesson: fluctuation in final number of detected PMT electrons S is larger by a factor of 2 than if σ_X^2 were zero.

12.3 Post-Scattering Energy Distribution of Fast Neutrons

Determine the energy distribution of a fast neutron with initial energy $E \gg k_B T$ after a single scattering from the nucleus of an element with mass number A.

Working in a classical, non-relativistic regime, consider a neutron with velocity v_1 and mass m incident on a stationary nucleus with mass $M \approx A \cdot m$. The velocity of the neutron-nucleus system's center of mass is

$$v^* = \frac{mv_1}{m+M} = \frac{mv_1}{m+Am} = \frac{v_1}{A+1}, \tag{12.6}$$

while the neutron and nucleus's velocities in the center of mass frame are

$$v_1^* = v_1 - v^* = \frac{A}{A+1} v_1 \qquad \text{and} \qquad v_2^* = 0 - v^* = -\frac{v_1}{A+1} \tag{12.7}$$

In the center of mass frame, conservation of energy and momentum produce the post-collision relationships

$$\left|\mathbf{v}_1'^*\right| = \left|\mathbf{v}_1^*\right| \qquad \text{and} \qquad \left|\mathbf{v}_2'^*\right| = \left|\mathbf{v}_2^*\right|.$$

In other words, the magnitudes of each particles' CoM-frame velocity are preserved; only the velocity directions change.

Transforming to the lab frame, the neutron's velocity after collision is

$$\mathbf{v}_1' = \mathbf{v}^* + \mathbf{v}_1'^*.$$

Using $\mathbf{v}_1'$, we aim to find the neutron's energy after the collision. Since $E \propto v_1'^2$, we first compute

$$v_1'^2 = (\mathbf{v}^* + \mathbf{v}_1'^*)(\mathbf{v}^* + \mathbf{v}_1'^*) = (v^*)^2 + 2v^* v_1^* \cos\theta' + (v_1^*)^2,$$

where θ is the angle between initial and final neutron direction in the center-of mass frame. We then substitute in v^* and v_1^* from Equations 12.6 and 12.7 to get

$$v_1'^2 = \frac{v_1^2}{(A+1)^2} + 2\frac{v_1}{A+1}\frac{Av_1}{A+1}\cos\theta + \frac{A^2 v_1^2}{(A+1)^2} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2} v_1^2.$$

The ratio of the neutron's pre- and post-collision energy is then

$$\frac{E'}{E} = \frac{v_1'^2}{v_1^2} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2}.$$

Since $\cos\theta \in [-1, 1]$, the maximum and minimum post-scattering energies are

$$E'_{\text{max}} = E \qquad \text{and} \qquad E'_{\text{min}} = \frac{(A-1)^2}{(A+1)^2} E \equiv \alpha E, \tag{12.8}$$

where we have defined the coefficient

$$\alpha \equiv \frac{(A-1)^2}{(A+1)^2}. \tag{12.9}$$

Note that maximum post-collision energy, which occurs when $\cos\theta = 1 \implies \theta = 0$, corresponds to no scattering at all, while minimum energy, when $\theta = \pi$, corresponds to perfect neutron back-scattering.

Energy Distribution

For review, the neutron's post-collision energy depends on scattering angle as

$$\frac{E'}{E} = \frac{v_1'^2}{v_1^2} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2}.$$

Plan: find the neutron's post-collision energy distribution from the distribution of the scattering angle θ using the chain rule

$$\frac{\mathrm{d}P}{\mathrm{d}E'} = \frac{\mathrm{d}P}{\mathrm{d}[\cos\theta]} \cdot \frac{\mathrm{d}[\cos\theta]}{\mathrm{d}E'}$$

We first compute

$$\frac{\mathrm{d}[\cos\theta]}{\mathrm{d}E'} = \frac{(A+1)^2}{2A}\frac{1}{E} \implies \frac{\mathrm{d}P}{\mathrm{d}E'} = \frac{\mathrm{d}P}{\mathrm{d}[\cos\theta]} \cdot \frac{(A+1)^2}{2A}\frac{1}{E}.$$

The distribution of $\cos\theta$ is found from scattering cross section via

$$\frac{\mathrm{d}P}{\mathrm{d}[\cos\theta]} = \frac{1}{\sigma_{\text{tot}}} \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\phi,$$

where the integral runs over only azimuthal angle ϕ because θ dependence is left in the $\cos\theta$ term. In practice, the neutron scattering cross section $\frac{d\sigma}{d\Omega}$ is complicated; for simplicity, we will use the approximation of isotropic scattering

$$\frac{d\sigma}{d\Omega} = \frac{\sigma_{\text{tot}}}{4\pi},$$

in which scattering is independent of azimuthal angle. This is a good approximation for low-energy neutrons scattering from light nuclei. Assuming $\frac{d\sigma}{d\Omega} = \frac{\sigma_{\text{tot}}}{4\pi}$, we have

$$\frac{dP}{d[\cos\theta]} = \frac{1}{\sigma_{\text{tot}}} \cdot \frac{\sigma_{\text{tot}}}{4\pi} \cdot \int d\phi = \frac{2\pi}{4\pi} = \frac{1}{2}.$$

The distribution of neutron post-collision energy is then

$$\frac{dP}{dE'} = \frac{dP}{d[\cos\theta]}\frac{d[\cos\theta]}{dE'} = \frac{1}{2} \cdot \frac{(A+1)^2}{2A}\frac{1}{E} = \frac{1}{1-\alpha}\frac{1}{E},$$

where we have used α from Equation 12.9. Finally, we recall the bounds on neutron energy from Equation 12.8 to get the complete result

$$\frac{dP}{dE'} = \begin{cases} 0 & E' < \alpha E \\ \frac{1}{(1-\alpha)E} & E' \in (\alpha E, E) \\ 0 & E' > E. \end{cases} \tag{12.10}$$

To summarize: this is the distribution of a fast neutron's ($E \gg k_B T$) post-collision energy E' after scattering from a single nucleus with mass number given by α via Equation 12.9. Note the result is just a uniform distribution, because of the assumption of uniform scattering.

13 Thirteenth Exercise Set

13.1 Slowing Neutrons to Thermal Energy

Determine the number of collisions N required for a fast neutron with energy $E_0 = 2\,\mathrm{MeV}$ to slow down to the thermal energy $E_\mathrm{T} = 1\,\mathrm{eV}$ in a hypothetical abosrber material with mass number $A = 10$.

From Equation 12.10 in the previous exercise set, a fast neutron's ($E \gg k_B T$) post-collision energy E' after scattering from a single nucleus with mass number A is distributed as

$$\frac{\mathrm{d}P}{\mathrm{d}E'} = \begin{cases} 0 & E' < \alpha E \\ \frac{1}{(1-\alpha)E} & E' \in (\alpha E, E) \\ 0 & E' > E \end{cases}, \quad \text{where } \alpha = \frac{(A-1)^2}{(A+1)^2}.$$

Goal is to find required number of collisions neutron to slow down to energy E_k. Principle is to first slow neutrons, then detect them.

We first define the dimensionless parameter ξ, which encodes the average energy lost by a neutron in a single collision, according to

$$\xi \equiv \left\langle \ln \frac{E_0}{E'} \right\rangle \implies \ln \frac{E'}{E_0} = -\xi \implies E' = E_0 e^{-\xi}, \tag{13.1}$$

where we have dropped the expectation signs after the first equality for conciseness.

After one collision, a neutron's energy decreases to $E'_1 = E_0 e^{-\xi}$, after the second collision to $E'_2 = E'_1 e^{-\xi} = E_0 e^{-2\xi}$, and so on... A neutron's energy after the N-th collision is then

$$E'_N = e^{-N\xi} E_0.$$

To find the problem's required number of collisions N to slow a neutron to thermal energy E_T, we set $E'_N \equiv E_T$ and get

$$E_\mathrm{T} = E^{-N\xi} E_0 \implies N = \frac{1}{\xi} \ln \frac{E_0}{E_\mathrm{T}}.$$

We now return to computing ξ in Equation 13.1, which we find with the distribution of neutron post-collision energy $\frac{\mathrm{d}P}{\mathrm{d}E'}$:

$$\xi \equiv \left\langle \ln \frac{E_0}{E'} \right\rangle = \int_{E'_\mathrm{min}}^{E'_\mathrm{max}} \ln \frac{E_0}{E'} \frac{\mathrm{d}P}{\mathrm{d}E'} \,\mathrm{d}E' = \frac{1}{(1-\alpha)E_0} \int_{\alpha E_0}^{E_0} \ln \frac{E_0}{E'} \,\mathrm{d}E',$$

where we have substituted in $E'_\mathrm{min} = \alpha E_0$ and $E'_\mathrm{max} = E_0$ from Equation 12.8. We solve the integral by defining the new variable $u \equiv E'/E_0 \implies \mathrm{d}u = (\mathrm{d}E')/E_0$, which produces the expression

$$\xi = \frac{1}{1-\alpha} \int_\alpha^1 \ln u \,\mathrm{d}u = -\frac{1}{1-\alpha} \Big[u\big(\ln u - 1\big) \Big]_\alpha^1 = 1 + \frac{\alpha}{1-\alpha} \ln \alpha.$$

We then substitute in $\alpha = \left(\frac{A-1}{A+1}\right)^2$ from Equation 12.9 to get

$$\xi = 1 + \frac{(A-1)^2}{2A} \ln \frac{A-1}{A+1}.$$

For heavy nuclei with $A \gg 1$, we can simplify ξ with the Taylor expansion

$$\ln \frac{x-1}{x+1} \sim 2\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{x^5} + \cdots\right),$$

which produces, up to first order in A, the result

$$\xi \approx 1 - \frac{A^2 - 2A - 1}{A^2} \sim \frac{2}{A}.$$

Without derivation a better approximation for ξ, which is exact up to one percent of the true value for $A > 10$, is

$$\xi \approx \frac{2}{A + 2/3}.$$

With ξ known, we find the average number of collisions N for thermalization from

$$N = \frac{1}{\xi} \ln \frac{E_0}{E_{\mathrm{T}}} = \frac{A + 2/3}{2} \ln \frac{2\,\mathrm{MeV}}{1\,\mathrm{eV}} = \frac{10 + 2/3}{2} \ln \frac{2\,\mathrm{MeV}}{1\,\mathrm{eV}} \approx 80.$$

Note: Neutron Absorption in Mixture Materials

For materials containing multiple elements, we generalize ξ to the average value

$$\bar{\xi} = \frac{\sum_i \sigma_i \xi_i}{\sum_i \sigma_i},$$

where σ_i is the cross section for neutron scattering in the i-th element of the absorber.

The table below shows values the number of collisions N required to slow a single neutron of energy 2 MeV to 1 eV.

Material	A	ξ	N
Hydrogen H	1	1	15
Deuterium D	2	0.75	20
Water H_2O	-	0.5	29
Iron Fe	56	0.035	414

13.2 Probability for Interaction in a ^{3}He Neutron Detector

A neutron detector consists of a cylindrical ionization chamber with radius $R = 4\,\mathrm{cm}$ *filled helium-3 gas at pressure of the order* $p \sim 5p_0$*. The cross section for neutron interaction with helium-3 in the experiment is known to be* $\sigma = 5 \cdot 10^3\,\mathrm{b}$*. The detector works on the basis of the neutron interaction*

$$^3_2\mathrm{He} + {}^1_0n \to {}^3_1\mathrm{H} + {}^1_1p + Q, \tag{13.2}$$

where $Q = 0.746\,\mathrm{MeV}$ *is the energy released by the reaction. We then indirectly detect the presence of neutrons from the released energy* Q*.*

Determine (i) the kinetic energies T_H and T_p of the hydrogen and proton products from a single reaction and (ii) the probability that a single neutron interacts when the detector when passing through.

Kinetic Energy of Products

The energy Q released in the reaction in Equation 13.2 comes from the kinetic energies of the resulting hydrogen and proton, i.e.

$$Q = T_\mathrm{H} + T_\mathrm{p}.$$

We then apply conservation of momentum (assuming a non-relativistic regime in which $p = 2mT$) to get

$$p_\mathrm{p} = p_\mathrm{H} \implies 2m_\mathrm{p}T_\mathrm{p} = 2m_\mathrm{H}T_\mathrm{H}$$

From here we find that

$$T_\mathrm{p} = \frac{m_\mathrm{H}}{m_\mathrm{p}}T_\mathrm{H} = \frac{m_\mathrm{H}}{m_\mathrm{p} + m_\mathrm{H}}Q \qquad \text{and} \qquad T_\mathrm{H} = Q - T_\mathrm{p} = \frac{m_\mathrm{p}}{m_\mathrm{p} + m_\mathrm{H}}Q$$

Noting that $^3_1\mathrm{H}$ and ^1_1p have masses $m_\mathrm{p} \approx 938\,\mathrm{MeV}$ and $m_\mathrm{H} \approx 2809\,\mathrm{MeV}$, the proton and hydrogen kinetic energies come out to

$$T_\mathrm{H} = 191\,\mathrm{keV} \qquad \text{and} \qquad T_\mathrm{p} = 573\,\mathrm{keV}.$$

Neutron-Helium Interaction Probability

We first find the mean neutron interaction distance λ in the helium absorbed using the general formula

$$\lambda = \frac{1}{n_\mathrm{s}\sigma},$$

where n_s is volume density of scatterers and σ is interaction cross section.

From thermodynamics, we recall ideal gas at standard temperature and pressure occupies a volume $V_\mathrm{m} = 22.4\,\mathrm{L\,mol^{-1}}$, from which we estimate the scatterer density n_s in the $^3_2\mathrm{He}$ detector at pressure $p = 5p_0$ as

$$n_\mathrm{s} = \frac{N_A}{V_\mathrm{m}} \cdot \frac{p}{p_0} = 5 \cdot \frac{6.022 \cdot 10^{23}\,\mathrm{mol^{-1}}}{22.4\,\mathrm{L\,mol^{-1}}} \approx 1.3 \cdot 10^{20}\,\mathrm{cm^{-3}}.$$

Using n_s, we then compute the neutron interaction distance λ:

$$\lambda = \frac{1}{n_\mathrm{s}\sigma} = \frac{1}{(1.3 \cdot 10^{20}\,\mathrm{cm^{-3}}) \cdot (5 \cdot 10^3\,\mathrm{b})} \approx 1.5\,\mathrm{cm}.$$

The probability P for single neutron to interact with the $R = 4\,\mathrm{cm}$ detector is then

$$P = \int_0^R \frac{1}{\lambda}e^{-\frac{x}{\lambda}}\,\mathrm{d}x = 1 - e^{-\frac{R}{\lambda}} = 1 - e^{-\frac{4\,\mathrm{cm}}{1.5\,\mathrm{cm}}} \approx 0.93.$$

13.3 Theory: Cherenkov Radiation

13.3.1 Threshold Speed and Momentum

TODO: parts of this section may need clarification.
A charged particle moving through material with refractive index n at speed $v > c_0/n$ will emit Cherenkov radiation, where c_0 is the speed of light in vacuum.

Geometry and coordinate system: (Best to see a picture).

Consider a particle moving along the positive x axis at speed $v > c_0/n$. Draw Cherenkov radiation wavefronts like a shock wave cone around the x axis, vertex at the instantaneous particle position, and spreading backwards along the x axis.

The angle between the normal to cone surface and the x axis is called the *Cherenkov angle* and is denoted by θ_C.

In time t, light travels a distance $s_\mathrm{light} = (c_0/n)t$ from a point a distance $x = vt$ behind the instantaneous particle position on the x axis to a point on the cone surface (along a path normal to the cone line). This consideration leads to

$$\cos\theta_\mathrm{C} = \frac{s_\mathrm{light}}{x} = \frac{(c_0/n)t}{vt} = \frac{1}{n\beta}, \quad \text{where } \beta \equiv \frac{v}{c_0}.$$

The Cherenkov angle is thus given by

$$\cos\theta_\mathrm{C} = \frac{1}{n\beta} \implies \theta_\mathrm{C} = \cos^{-1}\frac{1}{n\beta}. \tag{13.3}$$

The Cherenkov angle (i.e. the inverse cosine) is defined only for $1/(n\beta) < 1$, which gives a threshold speed

$$\frac{1}{n\beta} < 1 \implies \beta > \frac{1}{n} \qquad \text{(threshold for Cherenkov radiation)}.$$

In terms of momentum, using $\gamma\beta = (pc)/(mc^2)$, the Cherenkov threshold reads

$$\beta = \frac{pc}{\gamma mc^2} = \frac{pc}{E} = \frac{pc}{\sqrt{m^2c^4 + p^2c^2}} > \frac{1}{n} \implies pc > \frac{mc^2}{\sqrt{1 - 1/n^2}}. \tag{13.4}$$

13.3.2 Cherenkov Photon Frequency Spectrum

Without derivation, we quote that the Maxwell equations lead to a radiated Cherenkov photon energy spectrum (per unit length traveled by a radiating particle through detector material) given by

$$\frac{\mathrm{d}^2 E}{\mathrm{d}x\,\mathrm{d}\omega} = z^2\frac{\alpha\hbar\omega}{c}\sin^2\theta_\mathrm{C}, \tag{13.5}$$

where $\alpha = 1/137$ is the fine structure constant, z is the incident particle's charge in units of e_0, and ω is frequency of radiated energy.

Using the photon energy formula $E_\gamma = \hbar\omega$, the energy of N Cherenkov photons at the frequency ω is

$$E = N\hbar\omega \implies N = \frac{E}{\hbar\langle\omega\rangle} \implies \frac{\mathrm{d}N}{\mathrm{d}E} = \frac{1}{\hbar\langle\omega\rangle}.$$

Using $\frac{\mathrm{d}N}{\mathrm{d}E}$, we then convert Equation 13.5 to a distribution over N via

$$\frac{\mathrm{d}^2 N}{\mathrm{d}x\,\mathrm{d}\omega} = \frac{\mathrm{d}^2 E}{\mathrm{d}x\,\mathrm{d}\omega}\frac{\mathrm{d}N}{\mathrm{d}E} = \left(z^2\frac{\alpha\hbar\omega}{c}\sin^2\theta_\mathrm{C}\right)\cdot\frac{1}{\hbar\omega} = \frac{z^2\alpha}{c}\sin^2\theta_\mathrm{C}.$$

Finally, we express $\sin^2\theta_\mathrm{C}$ from Equation 13.3 and the use the identity $\sin^2 x = 1-\cos^2 x$ to write the Cherenkov photon frequency spectrum as

$$\frac{\mathrm{d}^2 N}{\mathrm{d}x\,\mathrm{d}\omega} = \frac{z^2\alpha}{c}\sin^2\theta_\mathrm{C} = \frac{z^2\alpha}{c}\left(1-\frac{1}{(n\beta)^2}\right), \tag{13.6}$$

13.3.3 Cherenkov Photon Wavelength Spectrum

Finally, we aim to derive $\frac{\mathrm{d}^2 N}{\mathrm{d}x\,\mathrm{d}\lambda}$, the distribution of the number of radiated Cherenkov photons (per unit distance traveled) with respect to wavelength.

We begin with the wavelength-frequency relationship

$$\omega = 2\pi\frac{c}{\lambda} \implies \frac{\mathrm{d}\omega}{\mathrm{d}\lambda} = \frac{2\pi c}{\lambda^2},$$

which we then substitute into Equation 13.6 to get

$$\frac{\mathrm{d}^2 N}{\mathrm{d}x\,\mathrm{d}\lambda} = \frac{\mathrm{d}^2 N}{\mathrm{d}x\,\mathrm{d}\omega}\frac{\mathrm{d}\omega}{\mathrm{d}\lambda} = \frac{2\pi z^2\alpha}{\lambda^2}\left(1-\frac{1}{(n\beta)^2}\right). \tag{13.7}$$

Note that, more generally, the refractive index n is a function of wavelength (or frequency), so both this formula and Equation 13.6, which treat n as a constant, are not perfectly accurate.

13.4 Detected Photons in a Cherenkov Detector

A Cherenkov radiation detector uses $d = 2\,\mathrm{cm}$ *of water* $(n \approx 1.33)$ *as the radiating material. The detector is sensitive to light in the range* $\lambda_\mathrm{min} = 250\,\mathrm{nm}$ *to* $\lambda_\mathrm{max} = 800\,\mathrm{nm}$, *in which range the detector's quantum efficiency is constant and equal to* $\eta = 10\%$. *Find the number* N_det *of detected photons when a proton of momentum* $pc = 2\,\mathrm{GeV}$ *passes through the detector.*

In general, we would compute number of detected photons by integrating the radiated Cherenkov photon spectrum times the detector efficiency over the problem's relevant wavelength range, i.e.

$$N_\mathrm{det} = d\int_{\lambda_\mathrm{min}}^{\lambda_\mathrm{max}} \eta(\lambda)\frac{\mathrm{d}^2 N}{\mathrm{d}x\,\mathrm{d}\lambda}\,\mathrm{d}\lambda.$$

However, because (for simplicity) the detector efficiency in this problem is constant, we can use the simplified expression

$$N_\mathrm{det} = \eta N_\mathrm{C},$$

where N_C is the number of radiated Cherenkov photons.

We find the number of radiated photons N_C by integrating over the wavelength spectrum in Equation 13.7. This reads

$$\begin{aligned} N_C &= d \cdot \int_{\lambda_{\min}}^{\lambda_{\max}} \frac{d^2N}{dx\,d\lambda}\,d\lambda = d \cdot 2\pi z^2 \alpha \left(1 - \frac{1}{(n\beta)^2}\right) \int_{\lambda_{\min}}^{\lambda_{\max}} \frac{d\lambda}{\lambda^2} \\ &= 2\pi d \cdot z^2 \alpha \left(1 - \frac{1}{(n\beta)^2}\right) \cdot \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max}\lambda_{\min}}. \end{aligned}$$

Our problem involves a proton, which has $z = 1$. We find β from the given momentum via equation chain:

$$\gamma\beta = \frac{pc}{mc^2} \longrightarrow \gamma^2 = 1 + (\beta\gamma)^2 \longrightarrow \beta^2 = 1 - \frac{1}{\gamma^2}.$$

The calculations read

$$\gamma\beta = \frac{2\,\text{GeV}}{938\,\text{MeV}} \approx 2.13 \implies \gamma^2 = 1 + (2.13)^2 \approx 5.55 \implies \beta^2 = 1 - \frac{1}{5.55} \approx 0.82.$$

The number of radiated Cherenkov photons is then

$$\begin{aligned} N_C &= 2\pi d \cdot z^2 \alpha \left(1 - \frac{1}{(n\beta)^2}\right) \cdot \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max}\lambda_{\min}} \\ &= 2\pi \frac{2\,\text{cm}}{137} \cdot \left(1 - \frac{1}{(1.33)^2 \cdot 0.82}\right) \cdot \frac{800\,\text{nm} - 250\,\text{nm}}{(250\,\text{nm}) \cdot (800\,\text{nm})} \\ &\approx 780. \end{aligned}$$

The number of detector photons N_{det} is then

$$N_{\text{det}} = \eta N_C \approx 0.1 \cdot 780 = 78.$$

14 Fourteenth Exercise Set

14.1 A Threshold Cherenkov Radiation Detector

We wish to classify pions and kaons with a threshold Cherenkov detector using aerogel with refractive index $n = 1.05$. From calibration, we know the detector registers an average of 10 Cherenkov photons emitted per pion when tested with beam of pions with momentum $pc = 100\,\text{GeV}$. For which values of incident particle momentum p does the detection efficiency for kaon-pion classification obey $\epsilon_\pi > 0.9$?

14.1.1 Finding Momentum for a Given Pion Detection Efficiency

From Equation 13.7 in the previous exercise, the Cherenkov photon wavelength spectrum (per unit length a radiating particle travels through detector material) is

$$\frac{\mathrm{d}^2 N}{\mathrm{d}x\,\mathrm{d}\lambda} = \frac{2\pi z^2 \alpha}{\lambda^2}\left(1 - \frac{1}{(n\beta)^2}\right) = \frac{2\pi z^2 \alpha}{\lambda^2}\left(1 - \frac{1}{(n\beta)^2}\right).$$

We then integrate over the arbitrary wavelength range λ_1 to λ_2 to get

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}x} &= \int_{\lambda_1}^{\lambda_2} \frac{\mathrm{d}^2 N}{\mathrm{d}x\,\mathrm{d}\lambda}\,\mathrm{d}\lambda = 2\pi z^2 \alpha \left(1 - \frac{1}{(n\beta)^2}\right)\int_{\lambda_1}^{\lambda_2}\frac{\mathrm{d}\lambda}{\lambda^2}\\ &= 2\pi\alpha z^2\left(1 - \frac{1}{(n\beta)^2}\right)\frac{\lambda_2 - \lambda_1}{\lambda_2\lambda_1}\end{aligned}$$

For the purposes of this problem, it is enough to know the proportionality

$$\frac{\mathrm{d}N}{\mathrm{d}x} \propto \left(1 - \frac{1}{(n\beta)^2}\right) \implies N \propto \left(1 - \frac{1}{(n\beta)^2}\right). \tag{14.1}$$

Importantly, for particles with large momentum ($pc \gg mc^2$), β will approach 1, and the $1/(n\beta)^2$ factor in Equation 14.1 becomes constant. In this case the number of Cherenkov photons emitted by a particle saturates at some maximum value N_{max}.

We find N_{max} for our problem from the given information that $N = 10$ for pions with $pc = 100\,\text{GeV}$. Since $100\,\text{GeV} \gg m_\pi c^2 = 140\,\text{MeV}$, we can assume the photon saturation value for pions is $N_{\text{max}} \approx N(pc = 100\,\text{GeV}) = 10$.

Relating β and Emitted Photon Number

We now aim to find a relationship between a particle's β factor and the number of Cherenkov photons N radiated by the particle in the detector. We begin with Equation 14.1 and define a proportionality constant a according to

$$N \propto \left(1 - \frac{1}{n^2\beta^2}\right) \implies N \equiv a\left(1 - \frac{1}{n^2\beta^2}\right)$$

We can find a from the large-momentum saturation value $N_{\text{max}} = 10$ by setting $\beta \approx 1$ and $N = N_{\text{max}}$. This reads

$$N_{\text{max}} = a\left(1 - \frac{1}{1 \cdot n^2}\right) \implies a = \frac{N_{\text{max}}}{1 - 1/n^2}.$$

The number of Cherenkov photons thus depends on radiating particle speed β as

$$N = a\left(1 - \frac{1}{n^2\beta^2}\right).$$

Technically, Cherenkov photon creation is a random process subject to Poisson fluctuations, so the above formula only gives the expected value $\langle N\rangle$ and should read

$$\langle N\rangle = a\left(1 - \frac{1}{n^2\beta^2}\right).$$

We then solve for β to get

$$\beta = \frac{1}{n\sqrt{1 - \left(\langle N\rangle/a\right)}}. \tag{14.2}$$

Pion Efficiency Condition

Using a Poisson distribution with mean $\langle N\rangle$, the probability of one pion emitting k Cherenkov photons is

$$P(k) = e^{-\langle N\rangle}\frac{\langle N\rangle^k}{k!}. \tag{14.3}$$

We then formulate the problem's $\epsilon_\pi = 0.9$ detection efficiency condition as

$$P_\pi(0) = 0.1.$$

In other words, 10% of incident pions won't emit a photon (while 90% of pions will emit a photon and thus be detected). We then substitute $P(0) = 0.1$ with $k = 0$ into Equation 14.3 to get

$$P(0) = e^{-\langle N\rangle}\cdot 1 = 0.1 \implies \langle N\rangle = -\ln(0.1) = 2.3.$$

Interpretation: the detector will have a pion detection efficiency $\epsilon_\pi = 0.9$ when an incident pion emits an average of $\langle N\rangle = 2.3$ Cherenkov photons. (While for $\langle N\rangle > 2.3$ the pion detection efficiency would exceed 0.9.)

Solving for Pion β and Momentum

We now aim to solve Equation 14.2 for the pion β factor satisfying $\epsilon_\pi = 0.9$. Using $N_{\max} = 10$, $n = 1.05$ and $\langle N\rangle = 2.3$, β comes out to

$$a = \frac{N_{\max}}{1 - (1/n^2)} \approx 107.6 \implies \beta = \frac{1}{n\sqrt{1 - (\langle N\rangle/a)}} \approx 0.963.$$

We then solve for the corresponding pion momentum pc via

$$\beta = \frac{pc}{E} = \frac{pc}{\gamma mc^2} \implies pc = \beta\gamma mc^2 = \frac{\beta mc^2}{\sqrt{1-\beta^2}} \approx 0.50\,\mathrm{GeV},$$

where we have used $m_\pi = 140\,\mathrm{MeV}$. Takeaway: the detector will register pions with momentum $pc \geq 0.5\,\mathrm{GeV}$ with efficiency $\epsilon_\pi \geq 0.9$.

14.1.2 Classifying Kaons and Pions

What is the probality of misidentifying a kaon as a pion at an incident particle momentum $p = 2\,\text{GeV}$*, assuming we wish to maintain the pion detection efficiency* $\epsilon_\pi = 0.9$*?*

For orientation, we first compute the pion and kaon threshold momentum for Cherenkov radiation. From Equation 13.4, the threshold momentum value is

$$pc > \frac{mc^2}{\sqrt{1 - 1/n^2}}.$$

For $n = 1.05$ using $m_\pi c^2 \approx 140\,\text{MeV}$ and $m_K \approx 500\,\text{MeV}$, the pion and kaon threshold momenta come out to

$$p_0^\pi \approx 0.46\,\text{GeV} \qquad \text{and} \qquad p_0^K \approx 1.64\,\text{GeV}.$$

Note that $pc = 2\,\text{GeV}$ is above both the pion and kaon Cherenkov thresholds.

We then draw Poisson distributions $P(N)$ for probability of an incident particle emitting N Cherenkov photons, which are centered at $\langle N \rangle^\pi$ and $\langle N \rangle^K$, respectively

How (binary) classification works: determine a decision boundary N_d for which an unknown particle emitting $N > N_\text{d}$ photons is classified as a pion, and a particle emitted $N < N_\text{d}$ is classified as a kaon.

Our goal: determine the decision boundary N_d for pion detection efficiency obeys $\epsilon_\pi \geq 0.9$ and, for this decision boundary, compute how many kaons are wrongly identified as pions (i.e. how many kaons will emit $N > N_\text{d}$ photons).

Finding Decision Boundary

We aim to find the decision boundary N_d for which, if we classify all particles emitting $N > N_\text{d}$ Cherenkov photons as pions, the pion detection efficiency is $\epsilon_\pi = 0.9$

Our first step is to determine the average number of Cherenkov photons $\langle N \rangle_\pi$ emitted by a pion with momentum $pc = 2\,\text{GeV}$.

Approximation: assume the number of Cherenkov photons per pion is saturated at $pc = 2\,\text{GeV}$ and assume $\langle N \rangle_\pi \approx N_\text{max} = 10$ at $pc = 2\,\text{GeV}$ to avoid computing $\langle N \rangle_\pi$.

Alternatively, for the sake of completeness, we will also compute the accurate value of $\langle N \rangle_\pi$ at $pc = 2\,\text{GeV}$ using the equation

$$\langle N \rangle = a\left(1 - \frac{1}{n^2\beta^2}\right) = \frac{N_\text{max}}{1 - 1/n^2}\left(1 - \frac{1}{n^2\beta^2}\right).$$

We find β for a $2\,\text{GeV}$ pion according to

$$\beta^2 = \frac{p^2c^2}{m^2c^4 + p^2c^2} = \frac{(2\,\text{GeV})^2}{(140\,\text{MeV})^2 + (2\,\text{GeV})^2} \approx 0.995.$$

The average number of emitted Cherenkov photons is then

$$\langle N \rangle_\pi = \frac{N_\text{max}}{1 - 1/n^2}\left(1 - \frac{1}{n^2\beta^2}\right) = \frac{10}{1 - 1/(1.05)^2}\left(1 - \frac{1}{(1.05)^2 \cdot 0.995}\right) \approx 9.5$$

We see the approximation $\langle N\rangle_\pi \approx 10$ is good.

A pion's number of emitted Cherenkov photons is distributed according to a Poisson distribution with mean $\langle N\rangle_\pi = 9.5$, so the probability for an incident pion emitting k Cherenkov photons is

$$P_\pi(k) = e^{-\langle N\rangle_\pi}\frac{(\langle N\rangle_\pi)^k}{k!}.$$

The corresponding cumulative distribution function F_π is

$$F_\pi(K) \equiv \sum_{k=0}^{K} P_\pi(k) = e^{-\langle N\rangle_\pi}\sum_{k=0}^{K}\frac{(\langle N\rangle_\pi)^k}{k!}.$$

We aim to find the decision boundary $K = N_{\text{d}}$ for which $F(N_{\text{d}}) = 1 - \epsilon_\pi = 0.1$. Testing values of K one by one, and using $\langle N\rangle_\pi = 9.5$, we find

$$F_\pi(5) = e^{-\langle N\rangle_\pi}\cdot\left[\frac{\big(\langle N\rangle_\pi\big)^0}{0!} + \cdots + \frac{\big(\langle N\rangle_\pi\big)^5}{5!}\right] \approx 0.088 < 0.1$$

$$F_\pi(6) = e^{-\langle N\rangle_\pi}\cdot\left[\frac{\big(\langle N\rangle_\pi\big)^0}{0!} + \cdots + \frac{\big(\langle N\rangle_\pi\big)^6}{6!}\right] \approx 0.16 > 0.1.$$

We first cross $F_\pi = 0.1$ when $K = 6$, so we choose the decision boundary $N_{\text{d}} = 6$.

Finding Wrongly-Classified Kaons

For review: We just found the decision boundary $N_{\text{d}} = 6$. If we classify all particles emitting $N > N_{\text{d}}$ Cherenkov photons as pions and all particles emitting $N \leq N_{\text{d}}$ photons as kaons, we know (from the above construction of $N_{\text{d}} = 6$) that the pion efficiency condition $\epsilon_\pi = 0.9$.

The corresponding kaon efficiency will be

$$\epsilon_K = F_K(N_{\text{d}}) = e^{-\langle N\rangle_K}\cdot\left[\frac{(\langle N\rangle_K)^0}{0!} + \cdots + \frac{(\langle N\rangle_K)^6}{6!}\right].$$

To compute ϵ_K, we need to find the average number of Cherenkov photons emitted by a kaon with $pc = 2\,\text{GeV}$. We first find the kaon beta factor β_K using $m_K \approx 500\,\text{MeV}$, which comes out to

$$\beta^2 = \frac{p^2c^2}{m^2c^4 + p^2c^2} = \frac{(2\,\text{GeV})^2}{(500\,\text{MeV})^2 + (2\,\text{GeV})^2} \approx 0.941.$$

The average number of Cherenkov photons emitted by a $\beta^2 = 0.941$ kaon is then

$$\langle N\rangle_K = \frac{N_{\text{max}}}{1 - 1/n^2}\left(1 - \frac{1}{n^2\beta^2}\right) = \frac{10}{1 - 1/(1.05)^2}\left(1 - \frac{1}{(1.05)^2\cdot 0.941}\right) \approx 3.88.$$

The kaon detection efficiency is then

$$\epsilon_K = e^{-N_K}\cdot\left[\frac{(\langle N\rangle_K)^0}{0!} + \cdots + \frac{(\langle N\rangle_K)^6}{6!}\right] \approx 0.88,$$

and so the percentage of incorrectly detected kaons is

$$1 - \epsilon_K \approx 1 - 0.88 = 0.12.$$

14.2 Ring-Imaging Cherenkov Detector

We wish to classify pions and kaons with momentum $p = 4.5\,\mathrm{GeV}$ with a ring-imaging Cherenkov detector using an aerogel radiating material with refractive index $n = 1.05$. Assume that when passing through the detector at momentum $pc = 4.5\,\mathrm{GeV}$, both kaons an pions emit an average of $\langle N\rangle_\pi \approx \langle N\rangle_K \approx 10$ Cherenkov photons. The Cherenkov angle for a single photon is measured with resolution $\sigma_{\theta_\mathrm{C}} = 15\,\mathrm{mrad}$.

Find the Cherenkov angle decision boundary $\theta_C^{(\mathrm{d})}$ for which 2.3% of pions are incorrectly classified as kaons. Then find the number of kaons incorrectly classified as pions at this decision boundary.

We find the Cherenkov angle of photons emitted by a particle with momentum pc via

$$\langle \cos\theta_\mathrm{C}\rangle = \frac{1}{n\beta}, \quad \text{where } \beta^2 = \frac{p^2c^2}{m^2c^4 + p^2c^2}.$$

We assume the Cherenkov angles for photons emitted by kaons and pions are both normally distributed with the same standard deviation $\sigma_{\theta_\mathrm{C}}$ but different mean values i.e.

$$\frac{\mathrm{d}P^{(\pi)}}{\mathrm{d}\theta_\mathrm{C}} \sim \mathcal{N}\big(\langle\theta_\mathrm{C}\rangle_\pi, \sigma_{\theta_\mathrm{C}}\big) \qquad \text{and} \qquad \frac{\mathrm{d}P^{(K)}}{\mathrm{d}\theta_\mathrm{C}} \sim \mathcal{N}\big(\langle\theta_\mathrm{C}\rangle_K, \sigma_{\theta_\mathrm{C}}\big)$$

Pions are lighter than kaons, so pions will have larger β than kaons at fixed momentum p and thus $\langle\theta_\mathrm{C}\rangle_\pi > \langle\theta_\mathrm{C}\rangle_K$.

The pion and kaon β values at $pc = 4.5\,\mathrm{GeV}$ are

$$\beta_\pi^2 = \frac{p^2c^2}{m_\pi^2c^4 + p^2c^2} = \frac{(4.5\,\mathrm{GeV})^2}{(140\,\mathrm{MeV})^2 + (4.5\,\mathrm{GeV})^2} \approx 0.9990$$
$$\beta_K^2 = \frac{p^2c^2}{m_K^2c^4 + p^2c^2} = \frac{(4.5\,\mathrm{GeV})^2}{(500\,\mathrm{MeV})^2 + (4.5\,\mathrm{GeV})^2} \approx 0.9878.$$

The corresponding mean Cherenkov angles are

$$\langle\theta_\mathrm{C}\rangle_\pi = \cos^{-1}\frac{1}{n\beta_\pi} = \cos^{-1}\left(\frac{1}{1.05\cdot 0.9990}\right) = 0.3067\,\mathrm{rad}$$
$$\langle\theta_\mathrm{C}\rangle_K = \cos^{-1}\frac{1}{n\beta_K} = \cos^{-1}\left(\frac{1}{1.05\cdot 0.9878}\right) = 0.2686\,\mathrm{rad}$$

The Cherenkov angle standard deviation for a single photon is given as $\sigma_{\theta_\mathrm{C}} = 15\,\mathrm{mrad}$. The Cherenkov angle standard deviation for a *particle*, which emits an average of $\langle N\rangle = 10$ photons, improves to

$$\sigma_{\theta_\mathrm{C}}^{\text{particle}} = \frac{\sigma_{\theta_\mathrm{C}}}{\sqrt{\langle N\rangle}} = \frac{15\,\mathrm{mrad}}{\sqrt{10}} \approx 4.74\,\mathrm{mrad}.$$

Computing Probability Distribution Values

We aim to find decision boundary $\theta_\mathrm{C}^{(\mathrm{d})}$ such that only 2.3% of pions are incorrectly identified. We formulate this condition in terms of the standard normal distribution's cumulative distribution function $\Phi(x)$ as

$$\Phi\left(\frac{\theta_{\mathrm{C}}^{(\mathrm{d})} - \langle\theta_{\mathrm{C}}\rangle_{\pi}}{\sigma_{\theta_{\mathrm{C}}}^{\text{particle}}}\right) = 0.023.$$

We first find the x solving $\Phi(x) = 0.023$. This comes out to $x \approx -2.0$, which leads to

$$x = \frac{\theta_{\mathrm{C}}^{(\mathrm{d})} - \langle\theta_{\mathrm{C}}\rangle_{\pi}}{\sigma_{\theta_{\mathrm{C}}}^{\text{particle}}} = -2.0 \implies \theta_{\mathrm{C}}^{(\mathrm{d})} = \langle\theta_{\mathrm{C}}\rangle_{\pi} - 2\sigma_{\theta_{\mathrm{C}}}^{\text{particle}}.$$

We then substitute in numerical values to get the decision boundary

$$\theta_{\mathrm{C}}^{(\mathrm{d})} = \langle\theta_{\mathrm{C}}\rangle_{\pi} - 2\sigma_{\theta_{\mathrm{C}}}^{\text{particle}} = 0.3067\,\mathrm{rad} - 2 \cdot 4.73\,\mathrm{mrad} = 0.2972\,\mathrm{rad}.$$

The percentage of incorrectly classified kaons is the percentage of kaons emitting photons with Cherenkov angle $\theta_{\mathrm{C}} > \theta_{\mathrm{C}}^{(\mathrm{d})}$. Using the Gaussian CDF, this is

$$\Phi\left(\frac{\theta_{\mathrm{C}}^{(\mathrm{d})} - \langle\theta_{\mathrm{C}}\rangle_{K}}{\sigma_{\theta_{\mathrm{C}}}^{\text{particle}}}\right) = \Phi\left(\frac{0.2972\,\mathrm{rad} - 0.2686\,\mathrm{rad}}{4.73\,\mathrm{mrad}}\right) = \Phi(6.1) \approx 1.0.$$

In other words, essentially all kaons are correctly classified for the decision boundary $\theta_{\mathrm{C}}^{(\mathrm{d})} = 0.4208\,\mathrm{rad}$ at momentum $pc = 4.5\,\mathrm{GeV}$.

TODO: The result from class was $\Phi(2.77)$ which is more reasonable. The discrepancy may come from $\langle\theta\rangle_{\pi} - \langle\theta\rangle_{K} \approx 22.6\,\mathrm{mrad}$ in class, while I had computed $\langle\theta\rangle_{\pi} - \langle\theta\rangle_{K} \approx 38.1\,\mathrm{mrad}$.