Contents

Miscellaneous Useful Material

Constants

$N_\mathrm{A} = 6.022 \cdot 10^{23}\,\mathrm{mol}^{-1}$

$\alpha \approx 1/137$

$m_\mathrm{e}c^2 \approx 511\,\mathrm{keV}$

$m_\mathrm{p}c^2 \approx 938\,\mathrm{keV}$

$m_\mathrm{n}c^2 \approx 940\,\mathrm{MeV}$

$m_\pi \approx 140\,\mathrm{MeV}$

$m_\mathrm{K} \approx 500\,\mathrm{MeV}$

$K_\mathrm{BB} \approx 0.3\,\mathrm{MeV\,g^{-1}\,cm^2}$

$(\beta\gamma)_\mathrm{MIP} \approx 3.5, \qquad \beta_\mathrm{MIP} \approx 0.96$

$1\,\mathrm{torr} = 1\,\mathrm{mmHg} \approx 133\,\mathrm{Pa}$

$k_B T\big|_{T=300\,\mathrm{K}} \approx 0.025\,\mathrm{eV}$

Some Relationships from Special Relativity

$\beta \equiv v/c \qquad \gamma \equiv 1/\sqrt{1-\beta^2}$

$E^2 = m^2c^4 + p^2c^2 \qquad E = \gamma mc^2 \qquad E = T + mc^2$

$\gamma\beta = \frac{pc}{mc^2} \qquad \gamma^2 = 1 + (\beta\gamma)^2 \qquad \beta^2 = 1 - 1/\gamma^2$

$\beta^2 = \frac{p^2c^2}{m^2c^4 + p^2c^2}$

$mc^2 = \frac{p^2c^2 - T^2}{2T}$

Some Relationships from Chemistry

The number density n_a (n_e) of atoms (electrons) in material with density ρ, molar mass M_m and atomic number Z is...

$n_\mathrm{a} = \frac{\rho N_\mathrm{A}}{M_\mathrm{m}} \qquad \text{and} \qquad n_\mathrm{e} = Z n_\mathrm{a}$

Molar mass M_m and relative atomic mass A_r are related by...

$M_\mathrm{m} = A_\mathrm{r} M_\mathrm{u}$, where $M_\mathrm{u} \equiv 1\,\mathrm{g\,mol^{-1}}$

$n_\mathrm{a} \approx \frac{\rho N_\mathrm{A}}{A M_\mathrm{u}}$ (if $A_\mathrm{r} \approx A$)

Ideal Gas

$V_\mathrm{m} \approx 22.4\,\mathrm{L\,mol^{-1}}$ (molar volume of ideal gas at STP)

$p_0 = 1\,\mathrm{atm} \approx 101\,\mathrm{kPa}$ (atmospheric pressure)

$n = \frac{N_\mathrm{A}}{V_\mathrm{m}} \frac{p}{p_0}$ (number density of ideal gas molecules)

Statistics

Binomial distribution: the probability of n events occurring over the course of N trials, where the probability of an event occurring in a single trial is p, is given by

$P(n|N,p) = \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$

Poisson distribution: the probability of n independent random events occuring in the time interval T, where the probability for an event per unit time is λ, is

$P(n|\lambda, T) = \frac{(\lambda T)^n}{n!} e^{-\lambda T}$, or...
$P(n|\mu) = \frac{\mu^n e^{-\mu}}{n!}$, where $\mu \equiv \lambda T$

$\langle X \rangle = \sigma_X^2 = \lambda$ (if X is Poisson distributed with mean μ)

Error function and *standard normal distribution* CDF

$\Phi(x) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$

Statistical significance in signal/background classification

$n = \frac{N_\mathrm{sig}}{\sigma_\mathrm{bg}}$
N_sig is number of counted signal events
σ_b is fluctuatio in background events

$\overline{N}_\mathrm{sig} = st \qquad \overline{N}_\mathrm{bg} = bt$ (if rates s, b are known)

Interaction of Particles and Matter

Scattering Cross Section

Beam with flux F of incident particles on target; $\frac{\mathrm{d}N_\mathrm{s}}{\mathrm{d}\Omega}$ is number of particles scattered into solid angle $\mathrm{d}\Omega$ per unit time.
$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{F}\frac{\mathrm{d}N_\mathrm{s}}{\mathrm{d}\Omega} \qquad \sigma_\mathrm{tot}(E) = \iint_\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega$
$N_\mathrm{s}(\Omega) = F S n_\mathrm{t} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\delta x \qquad N_\mathrm{s} = F S n_\mathrm{t} \sigma_\mathrm{tot} \delta x$

$\mathrm{d}P_\mathrm{scat} = n_\mathrm{t}\sigma_\mathrm{tot}\,\mathrm{d}x$ (probability for scattering in region $\mathrm{d}x$)
$P(x + \mathrm{d}x) = P(x) \cdot (1 - \mathrm{d}P_\mathrm{scat}) = P(x)(1 - n_\mathrm{t}\sigma_\mathrm{tot}\,\mathrm{d}x)$
$P(x) = e^{-n_\mathrm{t}\sigma_\mathrm{tot}x}$ (probability for not scattering up to x)

$1\,\mathrm{b} = 10^{-24}\,\mathrm{cm}^2 = 10^{-28}\,\mathrm{m}^2$

Charged Particles in Matter

Charged ionizing particle (IP) of mass m and valence Z_p travels through material with atomic number Z_m and density ρ.
Assume IP is heavy ($m \gg m_\mathrm{e}$)
Energy loss occurs primarily because of inelastic collisions of IP with electrons in the material.

Bethe-Bloch Formula

Valid for $\beta\gamma \sim (0.5, 10^3)$

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{4\pi}{m_\mathrm{e}c^2} \cdot \frac{n_\mathrm{e}Z_\mathrm{p}^2}{\beta^2} \cdot \left(\frac{e_0^2}{4\pi\epsilon_0}\right)^2 \ln\left[\left(\frac{2m_\mathrm{e}c^2\gamma^2\beta^2}{Z_\mathrm{m}I_0}\right) - \beta^2\right]$$
$$= K \cdot \frac{\rho Z_\mathrm{m}}{A} \cdot \frac{Z_\mathrm{p}^2}{\beta^2}\left[\ln\left(\frac{2m_\mathrm{e}c^2\beta^2\gamma^2}{Z_\mathrm{m}I_0}\right) - \beta^2\right]$$

$K \approx 0.3\,\mathrm{MeV\,g^{-1}\,cm^2} \qquad I_0 \sim 10\,\mathrm{eV}$

Small β approximation (e.g. for $\beta\gamma \lesssim 1$) produces...

$-\frac{\mathrm{d}E}{\mathrm{d}x} \sim \beta^{-2} \sim T^{-1} \implies \frac{\mathrm{d}T}{\mathrm{d}x} = -\frac{k}{T}, \qquad k = -T_0 \frac{\mathrm{d}T}{\mathrm{d}x}\Big|_{T=T_0}$

In Polyatomic Substances

Example: for H_2O_4, $i \in \{\mathrm{H}, \mathrm{O}\}$, e.g. $a_\mathrm{H} = 2$, $a_\mathrm{O} = 4$
$Z \to Z_\mathrm{eff} = \sum_i a_i Z_i$
$A \to A_\mathrm{eff} = \sum_i a_i A_i$
$\ln I \to \ln I_\mathrm{eff} = \sum_i \frac{a_i Z_i \ln I_i}{Z_\mathrm{eff}}$
$\left(-\frac{\mathrm{d}E}{\mathrm{d}x}\right)_\mathrm{tot} = \sum_i w_i \frac{\mathrm{d}E}{\mathrm{d}x}\Big|_i, \qquad w_i = \frac{a_i A_i}{\sum_j a_j A_j}$
$\left(-\frac{\mathrm{d}E}{\mathrm{d}x}\right)_\mathrm{tot} = K \cdot \frac{\rho_\mathrm{tot}Z_\mathrm{eff}}{A_\mathrm{eff}} \cdot \frac{Z_\mathrm{p}^2}{\beta^2}\left[\ln\left(\frac{2m_\mathrm{e}c^2\beta^2\gamma^2}{I_\mathrm{eff}}\right) - \beta^2\right]$

Photons in Matter

Three processes: photoelectric effect, Compton scattering, and pair production

$\sigma_\gamma = \sigma_{\text{pe}} + Z \cdot \sigma_{\text{C}} + \sigma_{\text{pair}}$

$\sigma_{\text{pe}} \sim \frac{Z^n}{E_\gamma^{7/2}},\ n \underset{\sim}{\in} [4,5]$

$j(x) = j_0 e^{-\mu x} \qquad \mu_\gamma = n_{\text{a}}\sigma_\gamma = \frac{\rho N_{\text{A}}}{M_{\text{m}}}\sigma_\gamma \qquad \lambda_\gamma = 1/\mu_\gamma$

$\mu_{\text{tot}} = \sum_i w_i \mu_i = \sum_i \left(\frac{A_i}{\sum_j A_j}\right)\mu_i$ (polyatomic substances)

Compton Scattering

Incident and scattered γ energies: E_γ, E_γ'; θ scattering angle

$\frac{E_\gamma'}{E_\gamma} = \frac{1}{1+\alpha(1-\cos\theta)}, \qquad \alpha \equiv \frac{E_\gamma}{m_{\text{e}}c^2}$

$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{r_{\text{e}}^2}{2}\left(\frac{E_\gamma'}{E_\gamma}\right)^2\left[\frac{E_\gamma'}{E_\gamma} + \frac{E_\gamma}{E_\gamma'} - \sin^2\theta\right]$

$\sigma_{\text{C}} = \frac{8\pi r_{\text{e}}^2}{3}\left[\frac{1-2\alpha+1.2\alpha^2}{(1+2\alpha)^2}\right], \qquad r_{\text{e}} = \frac{1}{4\pi\epsilon_0}\frac{e_0}{m_{\text{e}}c^2} \sim 2.8\,\text{fm}$

$\frac{\mathrm{d}\sigma_{\text{C}}}{\mathrm{d}T} = \frac{\pi r_0^2}{m_{\text{e}}c^2\alpha^2}\left[2 + \frac{s^2}{\alpha^2(1-s)^2} + \frac{s}{1-s}\left(s - \frac{2}{\alpha}\right)\right], \quad s = T/E_\gamma$

Particle Detectors

Energy Resolution

For a particle depositing energy E_{dep} and producing N ion pairs in a detector with Fano factor F...

$\mathcal{R} \equiv \frac{\sigma_{E_{\text{dep}}}}{E_{\text{dep}}} = \frac{\sigma_N}{N}$

Particle passes through detector: $\sigma_N = \sqrt{N}$

Particle stops inside detector: $\sigma_N = \sqrt{FN}$

$N = \frac{E_{\text{dep}}}{w_{\text{i}}} \implies \mathcal{R} = \sqrt{\frac{w_{\text{i}}}{E_{\text{dep}}}}$ or $\mathcal{R} = \sqrt{\frac{F w_{\text{i}}}{E_{\text{dep}}}}$

Ionization-Based Detectors

Parallel-Plate Ionization Cell

Consider a parallel-plate cell with pressure p, spacing d, potential difference U and constant electric field $E = U/d$.

$\mathrm{d}W = qE\,\mathrm{d}x = \frac{qU}{d}\,\mathrm{d}x$ (work on a charge q)

$\mathrm{d}W_{\text{C}} = CU\,\mathrm{d}U$ (change in capacitor energy)

$\mathrm{d}W = \mathrm{d}W_{\text{C}} \implies \mathrm{d}U = \frac{q}{C}\frac{\mathrm{d}x}{d}$

$v_{\text{d}} = \frac{E\mu}{p}$ (drift velocity, mobility)

$\Delta U(t) = \frac{q}{C}\frac{\mu}{pd}Et$ (before all ions reach electrodes)

$\Delta U = \frac{Q}{C}$ (when total charge Q reaches electrodes)

Multiplication Factor

For an incident particle freeing N_0 primary ions, which in turn free an average of N secondary ions...

$M \equiv \frac{N}{N_0}$ (multiplication factor)

λ is electron mean free path for ionizing collisions

$\alpha \equiv 1/\lambda$ is probaility for ionization per distance traveled

$\mathrm{d}N = N\alpha\,\mathrm{d}x \implies N(x) = N_0 e^{\alpha x}$ (for N initial electrons)

$M(x) \equiv N/N_0 = e^{\alpha x}$ or $M = \exp\left(\int_{x_1}^{x_2}\alpha(x)\,\mathrm{d}x\right)$

In a cell at pressure p with electric field E...

$\alpha = Ape^{-\frac{Bp}{E}}$ (Townsend discharge model; A, B given)

Cylindrical Ionization Chamber

For a cylindrical chamber with outer radius R and anode wire radius r_0 at voltage U_0....

$E(r) = \frac{U_0}{\ln(R/r_0)}\frac{1}{r} \qquad \phi(r) = -\frac{U_0}{\ln(R/r_0)}\ln\frac{r}{r_0} \qquad C = \frac{2\pi\epsilon_0 L}{\ln(R/r_0)}$

$v_{\text{d}} = E\mu$ (drift velocity v_{d}, mobility μ)

Signal detection delay t_{sig} between ionization event at $r = r^*$ and primary electrons reaching anode wire is...

$t_{\text{sig}} = \frac{\ln(R/r_0)R^2}{2\mu_{\text{e}}U_0}\left[\left(\frac{r^*}{R}\right)^2 - \left(\frac{r_0}{R}\right)^2\right] \approx \frac{\ln(R/r_0)}{2\mu_{\text{e}}U_0}(r^*)^2$

Only secondary positive ions contribute appreciably to signal

$U(t) = -\frac{Q_{\text{i}}}{4\pi\epsilon_0 L}\ln\left[1 + \frac{\mu_{\text{i}}CU_0}{\pi\epsilon_0 L r_0^2}\cdot(t - t_{\text{sig}})\right] \equiv -\frac{Q_{\text{i}}}{4\pi\epsilon_0 L}\ln\left(1 + \frac{t - t_{\text{sig}}}{t_0}\right)$

$U(t) = -\frac{N_{\text{s}}e_0}{4\pi\epsilon_0 L}\begin{cases}\approx 0 & t < t_{\text{sig}}\\ \ln\left(1 + \frac{t-t_{\text{sig}}}{t_0}\right) & t_{\text{sig}} < t < t_{\text{sig}} + t_{\text{ion}}.\end{cases}$

$t_0 \equiv \frac{\pi\epsilon_0 L r_0^2}{\mu_{\text{i}}CU_0}, \qquad t_{\text{ion}} \approx \frac{\ln(R/r_0)}{2\mu_{\text{i}}U_0}R^2$

$N_{\text{s}} = MN_{\text{p}} = M\frac{E_{\text{dep}}}{w_{\text{i}}}$

Measuring Momentum

Use a central drift chamber with beamline axis $\hat{\mathbf{z}}$ and magnetic field $\mathbf{B} \approx B\,\hat{\mathbf{e}}_z$

For particle of charge q with trajectory curvature radius R...

$\frac{mv_{\text{T}}^2}{R} = qv_{\text{T}}B \implies p_{\text{T}} = qBR$ (very simplified)

$p_{\text{T}}c \approx (0.3qBR)\,\text{GeV}$ ...

... if q is measured in e_0, B in tesla and R in meters

Momentum resolution $\sigma_{p_{\text{T}}}$ if trajectory resolution is σ_x...

$\frac{\sigma_{p_{\text{T}}}}{p_{\text{T}}} \approx \frac{\sqrt{96}\sigma_x}{qBL^2}\cdot p_{\text{T}}$ (three points on trajectory)

$\frac{\sigma_{p_{\text{T}}}}{p_{\text{T}}} \approx \frac{\sigma_x}{qBL^2}\cdot\sqrt{\frac{720}{N+4}}\cdot p_{\text{T}}$ (N points on trajectory)

L is characteristic length of cylindrical drift chamber

Semiconducting Detectors

E_{v} is top of valence band

E_{c} is bottom of conduction band

$E_{\text{g}} \equiv E_{\text{c}} - E_{\text{v}}$ is band gap

$f(E) = \frac{1}{e^{\beta(E-\mu)}+1}$ (Fermi-Dirac distribution)

$g_{\text{c}}(E) \approx \frac{1}{2\pi^2}\left(\frac{2m_{\text{c}}^*}{\hbar^2}\right)^{3/2}\sqrt{|E - E_{\text{c}}|}$

$g_{\text{v}}(E) \approx \frac{1}{2\pi^2}\left(\frac{2m_{\text{v}}^*}{\hbar^2}\right)^{3/2}\sqrt{|E - E_{\text{v}}|}$

$n_{\text{c}} = \frac{1}{4}\left(\frac{2m_{\text{c}}^*k_BT}{\pi\hbar^2}\right)^{3/2}e^{-\beta(E_{\text{c}}-\mu)} \equiv N_{\text{c}}(T)e^{-\beta(E_{\text{c}}-\mu)}$

$p_{\text{v}} = \frac{1}{4}\left(\frac{2m_{\text{v}}^*k_BT}{\pi\hbar^2}\right)^{3/2}e^{-\beta(\mu-E_{\text{v}})} \equiv P_{\text{v}}(T)e^{-\beta(\mu-E_{\text{v}})}$

In instrinsic SC: $n_{\text{c}} = p_{\text{v}} \equiv n_{\text{i}} \implies n_{\text{i}}^2 = N_{\text{c}}P_{\text{v}}e^{-\beta E_{\text{g}}}$

$n_{\text{i}} = \frac{1}{4}\left(\frac{2k_BT\sqrt{m_{\text{e}}^*m_{\text{h}}^*}}{\pi\hbar^2}\right)^{3/2}e^{-\frac{\beta E_{\text{g}}}{2}}$

Resistivity, Conductivty, Current Density

Consider conductor of conductivity σ_{E} with number density n of charge carriers q and mobility μ moving at drift velocity v_{d} under external electric field E

$j = \sigma_{\text{E}}E$ and $j = nqv_{\text{d}}$

$v_{\text{d}} = \mu E$

$\rho_{\text{E}} \equiv \frac{1}{\sigma_{\text{E}}}; \qquad \rho_{\text{E}} = \frac{1}{nq\mu} \qquad \sigma_{\text{E}} = nq\mu$

$j = e_0 n_{\text{i}}(\mu_{\text{e}} + \mu_{\text{h}})E$ (in instrinsic SC)

$j_n \approx e_0 N_d \mu_e E, \quad j_p \approx e_0 N_a \mu_h E$ (in doped SC)

$\sigma_n \approx e_0 N_d \mu_e, \quad \sigma_p \approx e_0 N_a \mu_h$ (in doped SC)

p-n Junction
Join p- and n-type SCs with dopant densities N_a and N_d

Depletion region spans $x \in (-x_p, x_n)$

$N_a x_p = N_d x_n$ (conservation of charge)

$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\rho(x)}{\epsilon\epsilon_0}$ (Poisson equation for potential)

$$\rho(x) \approx \begin{cases} -e_0 N_a & x \in (-x_p, 0) \\ e_0 N_d & x \in (0, x_n) \end{cases}$$

$$\frac{d\phi}{dx} \approx \begin{cases} \frac{e_0 N_a}{\epsilon\epsilon_0}(x + x_p) & x \in (-x_p, 0) \\ -\frac{e_0 N_d}{\epsilon\epsilon_0}(x - x_n) & x \in (0, x_n). \end{cases}$$

$\phi(-x_p) \equiv 0\,\mathrm{V}, \quad V_0 \equiv \phi(x_n) - \phi(-x_p) = \phi(x_n)$

$V_0 = \frac{e_0}{2\epsilon\epsilon_0}\left(N_d x_n^2 + N_a x_p^2\right)$

$$\phi(x) = \begin{cases} \frac{e_0 N_a}{2\epsilon\epsilon_0}(x + x_p)^2 & x \in (-x_p, 0) \\ V_0 - \frac{e_0 N_d}{2\epsilon\epsilon_0}(x - x_n)^2 & x \in (0, x_n) \end{cases}$$

$x_n^2 = \frac{2\epsilon\epsilon_0 V_0}{e_0 N_d\left(1 + \frac{N_d}{N_a}\right)} \qquad x_p^2 = \frac{2\epsilon\epsilon_0 V_0}{e_0 N_d\left(1 + \frac{N_a}{N_d}\right)}$

$d_{pn} = x_n + x_p = \sqrt{\frac{2\epsilon\epsilon_0 V_0}{e_0}\frac{N_a + N_d}{N_a N_d}}$

$d_{pn} \approx x_n \approx \sqrt{\frac{2\epsilon\epsilon_0 V_0}{e_0 N_d}}$ (if $N_a \gg N_d$)

$d_{pn} \approx x_p \approx \sqrt{\frac{2\epsilon\epsilon_0 V_0}{e_0 N_a}}$ (if $N_d \gg N_a$)

$d_{pn}^{(b)} = d_{pn}^{(0)}\sqrt{1 + \frac{V_b}{V_0}}$ (with reverse bias voltage V_b)

Approximate Expressions Depletion Region Width
$\sigma_n \approx e_0 N_d \mu_e, \ \sigma_p \approx e_0 N_a \mu_h \implies \rho_n \approx \frac{1}{e_0 N_d \mu_e}, \ \rho_p \approx \frac{1}{e_0 N_a \mu_h}$

$d_{pn} \approx \sqrt{2\epsilon\epsilon_0 \rho_n \mu_e V_0}$ (if $N_a \gg N_d$)

$d_{pn} \approx \sqrt{2\epsilon\epsilon_0 \rho_p \mu_h V_0}$ (if $N_d \gg N_a$)

Using $\epsilon_{Si} \approx 12$ and $\epsilon_{Ge} \approx 16$ we get...

$d_{Si} \approx 0.53\sqrt{\rho_n V_0} \cdot \mu\mathrm{m}$ (if $N_a \gg N_d$)

$d_{Si} \approx 0.32\sqrt{\rho_p V_0} \cdot \mu\mathrm{m}$ (if $N_d \gg N_a$)

$d_{Ge} \approx 1.00\sqrt{\rho_n V_0} \cdot \mu\mathrm{m}$ (if $N_a \gg N_d$)

$d_{Ge} \approx 0.65\sqrt{\rho_p V_0} \cdot \mu\mathrm{m}$ (if $N_d \gg N_a$)
... assuming V_0 in volts and ρ in $\Omega\,\mathrm{cm}$

Signal Dynamics in a p-n Semiconducting Detector
Shift coordinate system so that $x_p \equiv 0$

Let x_0 denote initial position of electron-hole pair

$\tau_h \equiv \frac{\epsilon\epsilon_0}{e_0 \mu_h N_a}, \qquad \tau_e \equiv \frac{\mu_h}{\mu_e}\tau_h, \qquad t_e = \tau_h \frac{\mu_h}{\mu_e} \cdot \ln\frac{d_{pn}}{x_0}$

$Q_e(t) = +\frac{e_0}{d_{pn}} x_0 \left(1 - e^{\frac{\mu_e}{\mu_h}\frac{t}{\tau_h}}\right)$ (for $t < t_e$)

$Q_h(t) = -\frac{e_0}{d_{pn}} x_0 \left(1 - e^{-t/\tau_h}\right)$

$I_e(t) = \frac{dQ_e}{dt} = -\frac{e_0}{d_{pn}}\frac{x_0}{\tau_h}\frac{\mu_e}{\mu_h} e^{\frac{\mu_e}{\mu_h}\frac{t}{\tau_h}}$ (for $t < t_e$)

$I_h(t) = \frac{dQ_h}{dt} = \frac{e_0}{d_{pn}}\frac{x_0}{\tau_h} e^{-t/\tau_h}$

$I_0^h \equiv \frac{e_0}{d_{pn}}\frac{x_0}{\tau_h}, \qquad I_0^e \equiv -\frac{e_0}{d_{pn}}\frac{x_0}{\tau_e}$

$$U_e(t) = \frac{I_0^e R}{1 + (RC)/\tau_e} \begin{cases} e^{t/\tau_e} - e^{-\frac{t}{RC}} & t < t_e \\ \left(e^{t_e/\tau_e} - e^{-\frac{t_e}{RC}}\right) e^{-\frac{(t - t_e)}{RC}} & t > t_e. \end{cases}$$

$U_h(t) = \frac{I_0^h R}{1 - (RC)/\tau_h}\left(e^{-t/\tau_h} - e^{-\frac{t}{RC}}\right),$

Limit Cases of Electron Signal

$$U_e(t) \approx I_0^e R \begin{cases} e^{t/\tau_e} - e^{-\frac{t}{RC}} & t < t_e \\ \left(e^{t_e/\tau_e} - e^{-\frac{t_e}{RC}}\right) e^{-\frac{(t - t_e)}{RC}} & t > t_e \end{cases} \quad (RC \ll \tau_e)$$

$U_e(t) = \frac{I_0^e \tau_e}{C}\left(e^{t_e/\tau_e} - 1\right) e^{-\frac{(t - t_e)}{RC}} = \frac{Q_e(t_e)}{C} e^{-\frac{(t - t_e)}{RC}}$ $(RC \gg \tau_e)$

Position Measurement
Consider parallel silicon microstrips separated by *pitch* p

$\sigma_x = \frac{p}{\sqrt{12}}$ (when using one strip to measure position)

$\overline{x} = \frac{\sum_i Q_i x_i}{\sum_i Q_i}$ (using multiple strips to measure position)

$\sigma_{\overline{x}}^2 \propto p^2 \frac{\sum_j \sigma_{Q_j}^2}{\left(\sum_i Q_i\right)^2} = p^2 \frac{(\text{noise})^2}{(\text{signal})^2} = \frac{p^2}{\text{SNR}^2}$

Q_j is charge on j-th strip
$\sigma_{Q_j}^2$ is resolution of charge on j-th strip

Scintillating Detectors
Consider scintillator with time constant τ, emitting $Y \equiv \frac{dN}{dE}$ photons per unit absorbed energy and photodetector with efficiency η and multiplication factor M

$\eta \equiv E_{scint}/E_{dep}, \quad E_{scint} = N_{scint} h\nu = hc/\lambda$ (efficiency)

$N(t) = N_0 e^{-t/\tau}$ (number of scintillation photons)

We assume a fast photodetector, so $I(t)$ follows $N(t)$, i.e.

$I(t) = I_0 e^{-t/\tau}$ (photodetector current)

$Q = \eta e_0 M Y E_{dep}$ (photodetector charge)

$Q = \int_0^\infty (t)\,dt = I_0 \tau \implies I_0 \tau = \eta e_0 M Y E_{dep}$

$U(t) = \frac{I_0 R}{1 - (RC)/\tau}\left(e^{-t/\tau} - e^{-\frac{t}{RC}}\right)$

$U(t) \approx \frac{I_0 \tau}{C} e^{-t/RC} = \frac{Q}{C} e^{-t/(RC)}$ $(RC \gg \tau)$

$U(t) \approx R I_0 e^{-t/\tau} = R I(t)$ $(RC \ll \tau)$

Fluctuations in Photomultipliers
X is the number of secondary electrons reaching PMT anode as a result of one initial cathode photoelectron

n is the number of initial cathode photoelectrons

S is the sum of all secondary electrons reaching PMT anode

n is Poisson-distributed with mean λ

$\langle S \rangle = \lambda \langle X \rangle$

$\sigma_S^2 = \lambda \left\langle X^2 \right\rangle \left(1 + \frac{\sigma_X^2}{\langle X \rangle^2}\right) \equiv F\lambda \left\langle X^2 \right\rangle$

Neutron Detection

In a material with scattering center density n_s and neutron cross section σ... $\lambda = \frac{1}{n_\mathrm{s}\sigma}$

In a material of width d with neutron MFP λ, probability for one neutron interaction is... $P = \int_0^d \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\,\mathrm{d}x$

Post-Scattering Energy Distribution of Fast Neutrons

Consider fast neutron with initial energy $E \gg k_B T$ scattering from a nucleus with mass number A at angle θ

Assume isotropic scattering $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\sigma_\mathrm{tot}}{4\pi}$

$$\alpha \equiv \frac{(A-1)^2}{(A+1)^2}$$

$$\frac{E'}{E} = \frac{A^2+2A\cos\theta+1}{(A+1)^2}$$

$$E'_\mathrm{max} = E, \quad E'_\mathrm{min} = \alpha E \qquad \text{(bounds on } E'\text{)}$$

$$\frac{\mathrm{d}P}{\mathrm{d}E'} = \begin{cases} 0 & E' < \alpha E \\ \frac{1}{(1-\alpha)E} & E' \in (\alpha E, E) \\ 0 & E' > E. \end{cases} \qquad \text{(distribution of } E'\text{)}$$

Slowing Neutrons to Thermal Energy

Goal: slow neutron from $E_0 \gg k_B T$ to $E_\mathrm{T} \sim k_B T$

$$\xi \equiv \left\langle \ln \frac{E_0}{E'} \right\rangle \implies \ln \frac{E'}{E_0} = -\xi \implies E' = E_0 e^{-\xi}$$

$$\xi = 1 + \frac{\alpha}{1-\alpha}\ln\alpha = 1 + \frac{(A-1)^2}{2A}\ln\frac{A-1}{A+1}$$

$$\xi \approx \frac{2}{A+2/3} \qquad \text{(for heavy nuclei, } A \gtrsim 10\text{)}$$

$$\overline{\xi} \equiv \frac{\sum_i \sigma_i \xi_i}{\sum_i \sigma_i} \qquad \text{(polyatomic materials)}$$

$$E'_N = e^{-N\xi} E_0 \qquad \text{(energy after } N\text{-th collision)}$$

$$N = \frac{1}{\xi}\ln\frac{E_0}{E_\mathrm{T}} \qquad \text{(collisions to reach energy } E_\mathrm{T}\text{)}$$

Cherenkov Radiation

Consider particle with charge $z = q/e_0$ moving along x axis in material with refractive index n at speed $v > c/n$

$$\cos\theta_\mathrm{C} = \frac{1}{n\beta} \implies \theta_\mathrm{C} = \cos^{-1}\frac{1}{n\beta} \qquad \text{(Cherenkov angle)}$$

$$\beta > 1/n \quad \text{or} \quad pc > \frac{mc^2}{\sqrt{1-(1/n^2)}} \qquad \text{(thresholds for radiation)}$$

$$\frac{\mathrm{d}^2 E}{\mathrm{d}x\,\mathrm{d}\omega} = z^2 \frac{\alpha\hbar\omega}{c}\sin^2\theta_\mathrm{C}$$

$$\frac{\mathrm{d}^2 N}{\mathrm{d}x\,\mathrm{d}\omega} = \frac{z^2\alpha}{c}\sin^2\theta_\mathrm{C} = \frac{z^2\alpha}{c}\left(1 - \frac{1}{(n\beta)^2}\right)$$

$$\frac{\mathrm{d}^2 N}{\mathrm{d}x\,\mathrm{d}\lambda} = \frac{2\pi z^2\alpha}{\lambda^2}\sin^2\theta_\mathrm{C} = \frac{2\pi z^2\alpha}{\lambda^2}\left(1 - \frac{1}{(n\beta)^2}\right)$$

Cherenkov Detectors

Consider a detector sensitive to radiation in the range $\lambda_\mathrm{min}, \lambda_\mathrm{max}$ with efficiency $\eta(\lambda)$

$$N_\mathrm{det} = d\int_{\lambda_\mathrm{min}}^{\lambda_\mathrm{max}} \eta(\lambda)\frac{\mathrm{d}^2 N}{\mathrm{d}x\,\mathrm{d}\lambda}\,\mathrm{d}\lambda$$

$$N_\mathrm{C} \propto \sin^2\theta_\mathrm{C} = \left(1 - \frac{1}{(\beta n)^2}\right) \implies N_\mathrm{C} \to N_\mathrm{max} \text{ as } \beta \to 1$$

$$\langle N \rangle = \frac{N_\mathrm{max}}{1-1/n^2}\left(1 - \frac{1}{\beta^2 n^2}\right) \equiv a\left(1 - \frac{1}{\beta^2 n^2}\right)$$

$$\implies \beta = \frac{1}{n\sqrt{1-\left(\langle N\rangle/a\right)}}$$